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An Unified Consumption and Production Model for a Closed Economy

Cătălin Angelo Ioan¹, Gina Ioan²

Abstract: The article presents an unified consumption and production model for a closed economy.

Keywords: consumption, production, utility

JEL Code: E17; E27

1. Introduction

Let consider n goods: G_1, \dots, G_n whose elasticity of utility is constant, their prices being p_1, \dots, p_n . For a consumer whose available income is V , the utility function corresponding to the consumption of x_p units of good G_p , $p = \overline{1, n}$: $U(x_1, \dots, x_n) = Ax_1^{\alpha_1} \dots x_n^{\alpha_n}$ where α_p is the elasticity of utility in relation to the good G_p , and A is a positive constant.

The issue of maximizing the utility relative to the restriction: $\sum_{k=1}^n p_k x_k \leq V$ is:

$$\begin{cases} \max U(x_1, \dots, x_n) \\ \sum_{k=1}^n p_k x_k \leq V \\ x_1, \dots, x_n \geq 0 \end{cases}$$

Considering the Lagrangeian:

$$\Phi(x_1, \dots, x_n, \lambda) = U(x_1, \dots, x_n) + \lambda \left(\sum_{k=1}^n p_k x_k - V \right)$$

the maximum condition with restrictions must satisfy:

$$\begin{cases} \frac{\partial \Phi}{\partial x_j} = \frac{\partial U}{\partial x_j} + \lambda p_j = 0, j = \overline{1, n} \\ \frac{\partial \Phi}{\partial \lambda} = \sum_{j=1}^n p_j x_j - V = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_j A x_1^{\alpha_1} \dots x_j^{\alpha_j-1} \dots x_n^{\alpha_n} + \lambda p_j = 0, j = \overline{1, n} \\ \sum_{j=1}^n p_j x_j - V = 0 \end{cases} \Leftrightarrow$$

$$-\sum_{j=1}^n \frac{\alpha_j A x_1^{\alpha_1} \dots x_j^{\alpha_j-1} \dots x_n^{\alpha_n}}{\lambda} x_j - V = 0 \Leftrightarrow \frac{A x_1^{\alpha_1} \dots x_j^{\alpha_j} \dots x_n^{\alpha_n}}{\lambda} \sum_{j=1}^n \alpha_j + V = 0 \Leftrightarrow$$

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$$\lambda = -\frac{Ax_1^{\alpha_1} \dots x_j^{\alpha_j} \dots x_n^{\alpha_n} \sum_{j=1}^n \alpha_j}{V}.$$

By re-replacing, we get the optimal solution to the problem:

$$x_j^* = \frac{\alpha_j V}{p_j \sum_{j=1}^n \alpha_j}, j = \overline{1, n}$$

Let us also consider a producer with a number of K capital units, having the price of p_K and L workers whose hourly wage is w for a working time t . If the elasticity of production in relation to capital and labor are constant, the function of production is: $Q(t, K, L) = CK^\beta L^\gamma$ where β and γ are the elasticities of production in relation to the capital, respectively the labor, C being a constant.

Total cost of production: $CT = p_K K + twL$ leads to a gross profit corresponding to a sales price p : $\pi(t, K, L) = pQ(t, K, L) - CT = CpK^\beta L^\gamma - p_K K - twL$.

For a given production Q_0 , the profit maximization condition returns to minimizing the total cost, so to

$$\text{the problem: } \begin{cases} \min(p_K K + twL) \\ CK^\beta L^\gamma = Q_0 \\ K, L \geq 0 \end{cases}.$$

Considering the Lagrangeian:

$$\Phi(x_1, \dots, x_n, \lambda) = p_K K + twL + \lambda(CK^\beta L^\gamma - Q_0)$$

the minimum condition with restrictions must satisfy:

$$\begin{cases} \frac{\partial \Phi}{\partial K} = p_K + \lambda \beta CK^{\beta-1} L^\gamma = 0 \\ \frac{\partial \Phi}{\partial L} = tw + \lambda \gamma CK^\beta L^{\gamma-1} = 0 \\ \frac{\partial \Phi}{\partial \lambda} = CK^\beta L^\gamma - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda = -\frac{p_K}{\beta CK^{\beta-1} L^\gamma} \\ \lambda = -\frac{tw}{\gamma CK^\beta L^{\gamma-1}} \\ CK^\beta L^\gamma - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{p_K}{\beta CK^{\beta-1} L^\gamma} = \frac{tw}{\gamma CK^\beta L^{\gamma-1}} \\ CK^\beta L^\gamma - Q_0 = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} \frac{p_K}{\beta L} = \frac{tw}{\gamma K} \\ CK^\beta L^\gamma - Q_0 = 0 \end{cases} \Leftrightarrow \begin{cases} L^* = \frac{Q_0}{C} \left(\frac{p_K \gamma}{\beta tw} \right)^{\frac{\beta}{\beta+\gamma}} \\ K^* = \frac{Q_0}{C} \left(\frac{p_K \gamma}{\beta tw} \right)^{-\frac{\gamma}{\beta+\gamma}} \end{cases}, \text{ and the minimum total cost:}$$

$$CT^* = \frac{Q_0}{C} \left(p_K \left(\frac{p_K \gamma}{\beta tw} \right)^{-\frac{\gamma}{\beta+\gamma}} + tw \left(\frac{p_K \gamma}{\beta tw} \right)^{\frac{\beta}{\beta+\gamma}} \right).$$

The maximum profit is:

$$\pi^*(t, K^*, L^*) = pQ_0 - \frac{Q_0}{C} \left(p_K \left(\frac{p_K \gamma}{\beta t w} \right)^{-\frac{\gamma}{\beta+\gamma}} + t w \left(\frac{p_K \gamma}{\beta t w} \right)^{\frac{\beta}{\beta+\gamma}} \right)$$

2. The Model

Suppose there are a number of n firms: F_1, \dots, F_n each having a number of L_i employees, $i = \overline{1, n}$ where we will include, for simplification, the entrepreneur of the firm. Let w_i - the average hourly wage for F_i , t_i - the working time during the analysis period in F_i . We will also assume that F_i produces a single good (of constant elasticity): G_i whose sales price is p_i .

From the total revenue received, each employee pays a tax quota to the state budget χ .

For health insurance, pensions and other services that will later be paid back to employees, they pay a share ξ of the salary received. Let us consider the providers of these services (a single service G_{n+j} , $j = \overline{1, m}$ for each firm) as being the firms F_{n+1}, \dots, F_{n+m} each having L_{n+j} , $j = \overline{1, m}$ employees (including the entrepreneur), with w_{n+j} - the average hourly wage corresponding to the company F_{n+j} and t_{n+j} - the working time worked during the analysis period in F_{n+j} . The service price offered by F_{n+j} will be p_{n+j} , $j = \overline{1, m}$.

Therefore, the tax paid by each employee will be: $T_b = \chi w_i t_i$, $i = \overline{1, n+m}$ and for public services: $T_s = \xi w_i t_i$, $i = \overline{1, n+m}$.

The revenues available to F_i staff are therefore (for each individual employee): $V_i = (1 - \chi - \xi) w_i t_i$, $i = \overline{1, n+m}$.

On the other hand, the amount of salaries received by service providers comes from the share ξ therefore: $\xi \sum_{i=1}^{n+m} w_i t_i L_i = \sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j}$ or: $\sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j} = \frac{\xi}{1-\xi} \sum_{i=1}^n w_i t_i L_i$.

The entrepreneur of F_i , $i = \overline{1, n+m}$ will allocate the profits made for investments that will be considered as goods produced by firms. Let us consider the output of F_i as: $Q_i(t, K_i, L_i) = C_i K_i^{\beta_i} L_i^{\gamma_i}$ where β_i and γ_i represent the constantly assumed elasticities of production relative to K_i , respectively L_i , C_i - positive constant. At a price of capital p_{K_i} , the total cost of production in F_i becomes: $CT_i = p_{K_i} K_i + t_i w_i L_i$. Therefore, at a sale price p_i of the G_i asset, F_i 's profit is:

$$\pi_i = p_i Q_i(t, K_i, L_i) - p_{K_i} K_i - t_i w_i L_i = p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i$$

The F_i 's firm's entrepreneur income will therefore be just that π_i , $i = \overline{1, n+m}$.

Let considering the set S of social assistants (pensioners, people without income etc.) with a number of M people whose incomes represents a share ρ of the taxes paid by employees (the remainder being allocated to government consumption, public works etc.). Their income will therefore be:

$$\rho(TP_b + TS_b) = \rho \chi \sum_{i=1}^{n+m} w_i t_i L_i \text{ and the average income per person: } \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i.$$

In the following, we will consider that the utility function of any employee of a production, service or social assistance company will be the same for all consumers within the category (it may be different from company to company – as an example, the utility of books is different for employees of an educational establishment and another for meat producers). In addition, we will assume that all the production of a company will be sold.

Consider the utility functions for an employee of F_i : $\tilde{U}_i(x_{i1}, \dots, x_{i, n+m}) = A_i x_{i1}^{\alpha_{i1}} \dots x_{i, n+m}^{\alpha_{i, n+m}}$, $i = \overline{1, n+m}$ where x_{ij} represents the quantity of good G_j consumed by an employee of F_i and for social assistants: $\tilde{U}_S(x_1, \dots, x_{n+m}) = A x_1^{\alpha_1} \dots x_{n+m}^{\alpha_{n+m}}$ where x_j represents the amount of good G_j consumed by a social assistant.

The utility functions of the entrepreneurs in the investment activity will be: $\tilde{\tilde{U}}_i(y_{i1}, \dots, y_{in}) = B_i y_{i1}^{\delta_{i1}} \dots y_{in}^{\delta_{in}}$, $i = \overline{1, n+m}$ where y_{ij} represents the amount of good G_j consumed by the F_i 's entrepreneur.

Every employee and entrepreneurs want to maximize their utilities in the context of disposable income, so problems arise:

- $$\begin{cases} \max_{x_{i1}, \dots, x_{i, n+m}} \tilde{U}_i(x_{i1}, \dots, x_{i, n+m}) = A_i x_{i1}^{\alpha_{i1}} \dots x_{i, n+m}^{\alpha_{i, n+m}} \\ \sum_{k=1}^{n+m} p_k x_{ik} \leq (1 - \chi - \xi) w_i t_i \\ x_{i1}, \dots, x_{i, n+m} \geq 0 \end{cases} \quad \text{- for company employees } F_i, i = \overline{1, n+m};$$
- $$\begin{cases} \max_{x_1, \dots, x_{n+m}} \tilde{U}_S(x_1, \dots, x_{n+m}) = A x_1^{\alpha_1} \dots x_{n+m}^{\alpha_{n+m}} \\ \sum_{k=1}^{n+m} p_k x_k \leq \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i \\ x_{i1}, \dots, x_{i, n+m} \geq 0 \end{cases} \quad \text{- for social assistants;}$$
- $$\begin{cases} \max_{y_{i1}, \dots, y_{in}} \tilde{\tilde{U}}_i(y_{i1}, \dots, y_{in}) = B_i y_{i1}^{\delta_{i1}} \dots y_{in}^{\delta_{in}} \\ \sum_{k=1}^n p_k y_{ik} \leq p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i, i = \overline{1, n+m} \\ y_{i1}, \dots, y_{in} \geq 0 \end{cases} \quad \text{- for entrepreneurs.}$$

It follows from the above that the optimum quantities of products are:

- $$x_{ij}^* = \frac{\alpha_{ij} (1 - \chi - \xi) w_i t_i}{p_j \sum_{k=1}^{n+m} \alpha_{ik}}, j = \overline{1, n+m} \quad \text{- for company employees } F_i, i = \overline{1, n+m};$$
- $$x_j^* = \frac{\alpha_j \frac{\rho \chi}{M} \sum_{i=1}^{n+m} w_i t_i L_i}{p_j \sum_{k=1}^{n+m} \alpha_k}, j = \overline{1, n+m} \quad \text{- for social assistants;}$$
- $$y_{ij}^* = \frac{\delta_{ij} p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i}{p_j \sum_{k=1}^{n+m} \delta_{ik}}, j = \overline{1, n+m} \quad \text{- for entrepreneurs.}$$

Therefore, the amount of required G_j needed is:

$$Q_j = \sum_{i=1}^{m+n} L_i x_{ij}^* + M \sum_{i=1}^{m+n} x_j^* + \sum_{i=1}^{m+n} y_{ij}^* =$$

$$\frac{1-\chi-\xi}{p_j} \sum_{i=1}^{m+n} L_i \frac{\alpha_{ij} w_i t_i}{\sum_{k=1}^{n+m} \alpha_{ik}} + \frac{\rho \chi \alpha_j (m+n) \sum_{i=1}^{n+m} w_i t_i L_i}{p_j \sum_{k=1}^{n+m} \alpha_k} + \frac{1}{p_j} \sum_{i=1}^{m+n} \frac{\delta_{ij} p_i C_i K_i^{\beta_i} L_i^{\gamma_i} - p_{K_i} K_i - t_i w_i L_i}{\sum_{k=1}^{n+m} \delta_{ik}}, j = \overline{1, n+m}.$$

Returning to the problem of maximizing F_j 's profit for the quantity Q_j we have:

$$\begin{cases} \min (p_{K_j} K_j + t_j w_j L_j) \\ C_j K_j^{\beta_j} L_j^{\gamma_j} = Q_j \\ K_j, L_j \geq 0 \end{cases}, j = \overline{1, n+m}$$

from where:

$$\begin{cases} L_j^* = \frac{Q_j}{C_j} \left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\beta_j}{\beta_j + \gamma_j}} \\ K_j^* = \frac{Q_j}{C_j} \left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\gamma_j}{\beta_j + \gamma_j}} \end{cases}$$

Noting for simplicity:

$$s_i = w_i t_i, r_{ij} = \frac{\alpha_{ij} s_i}{\sum_{k=1}^{n+m} \alpha_{ik}}, u_j = \frac{\rho \chi (m+n) \alpha_j}{\sum_{k=1}^{n+m} \alpha_k}, v_{ij} = \frac{\delta_{ij} p_i C_i}{\sum_{k=1}^{n+m} \delta_{ik}}, z_i = \frac{p_{K_i}}{\sum_{k=1}^{n+m} \delta_{ik}}, f_i = \frac{s_i}{\sum_{k=1}^{n+m} \delta_{ik}},$$

$$g_j = \frac{\left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\beta_j}{\beta_j + \gamma_j}}}{C_j p_j} \Rightarrow \frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} = g_j^{\frac{\beta_j + \gamma_j}{\beta_j}} (C_j p_j)^{\frac{\beta_j + \gamma_j}{\beta_j}} \text{ from where: } \frac{\left(\frac{p_{K_j} \gamma_j}{\beta_j t_j w_j} \right)^{\frac{\gamma_j}{\beta_j + \gamma_j}}}{C_j p_j} = g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1}$$

we find that:

$$\begin{cases} L_j^* = g_j \left((1-\chi-\xi) \sum_{i=1}^{m+n} r_{ij} L_i^* + u_j \sum_{i=1}^{n+m} s_i L_i^* + \sum_{i=1}^{m+n} (v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - z_i K_i^* - f_i L_i^*) \right) \\ K_j^* = g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} \left((1-\chi-\xi) \sum_{i=1}^{m+n} r_{ij} L_i^* + u_j \sum_{i=1}^{n+m} s_i L_i^* + \sum_{i=1}^{m+n} (v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - z_i K_i^* - f_i L_i^*) \right) \end{cases}$$

or, in other words:

$$\begin{cases} L_j^* = \sum_{i=1}^{m+n} g_j (r_{ij} (1-\chi-\xi) + u_j s_i - f_i) L_i^* + \sum_{i=1}^{m+n} g_j v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} g_j z_i K_i^* \\ K_j^* = \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} (r_{ij} (1-\chi-\xi) + u_j s_i - f_i) L_i^* + \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} v_{ij} K_i^{*\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} z_i K_i^* \end{cases}$$

Noting again:

- $Y_{1,ij} = g_j (r_{ij}(1 - \chi - \xi) + u_j s_i - f_i)$
- $Y_{2,ij} = g_j v_{ij}$
- $Y_{3,ij} = g_j z_i$
- $Y_{4,ij} = g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} (r_{ij}(1 - \chi - \xi) + u_j s_i - f_i)$
- $Y_{5,ij} = g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} v_{ij}$
- $Y_{6,ij} = g_j^{\frac{\gamma_j}{\beta_j}} (C_j p_j)^{\frac{\gamma_j}{\beta_j} - 1} z_i$

we obtain:

$$\begin{cases} L_j^* = \sum_{i=1}^{m+n} Y_{1,ij} L_i^* + \sum_{i=1}^{m+n} Y_{2,ij} K_i^{\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} Y_{3,ij} K_i^* \\ K_j^* = \sum_{i=1}^{m+n} Y_{4,ij} L_i^* + \sum_{i=1}^{m+n} Y_{5,ij} K_i^{\beta_i} L_i^{*\gamma_i} - \sum_{i=1}^{m+n} Y_{6,ij} K_i^* \end{cases}, j = \overline{1, n+m}$$

The system solution will provide the optimal number of employees of each firm as well as the required capital.

On the other hand, provided that: $\sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j} = \frac{\xi}{1-\xi} \sum_{i=1}^n w_i t_i L_i$ follows:

$$\sum_{j=1}^m w_{n+j} t_{n+j} L_{n+j}^* = \frac{\xi}{1-\xi} \sum_{i=1}^n w_i t_i L_i^*.$$

By replacing the above optimal solutions, we obtain the link between the two quotas (the only ones that are imposed at government level): χ (tax) and ξ - for health insurance, pensions and other services.

3. References

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