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Interpretations of Hyperbolic Growth

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Abstract. Hyperbolic growth describes the historical economic growth and historical growth of population, but their mechanism remains unexplained. Presented here is a brief survey of attempts to understand hyperbolic growth. Mathematical formulations are in general complicated and there is no clear advantage in using them because they do not give better description of data than the simple, two-parameter hyperbolic formula. They also do not explain the mechanism of growth. The well-known simple formula suggests a simple explanation. Two examples show how two independent investigations were on a brink of making an important and breakthrough discovery and how their potential discovery was thwarted by the established knowledge in demography and in economic research. Researchers who could have used their expertise to suggests new research directions and to advance science were constrained by doctrines, which are widely accepted by faith.
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1. Introduction

Historical economic growth and the growth of population were hyperbolic (Nielsen, 2014, 2016a, 2016b, 2016c). Hyperbolic growth is described by an exceptionally simple mathematical formula. It is just the reciprocal of a linear function. Many attempts were made to understand hyperbolic growth or to give an alternative mathematical description. These descriptions or interpretations tend to be complicated, maybe because hyperbolic distributions appear to be complicated. Furthermore, they do not explain the mechanism of growth. They also do not give better description of data than the description furnished by the simple mathematical equation. We shall present here a few examples of earlier attempts to explain or to describe hyperbolic distributions.

2. Technology and the growth of population

Using correlations between two processes might be tempting in order to explain the mechanism of growth but correlations could be spurious and misleading. Just because there is a correlation between two processes it does not mean that one process influences another. It does not also mean that there is a cause-effect relation between two observed processes. One has to be on guard when using such correlations because they can lead easily to loops and to the incorrect interpretation of the mechanism of growth.

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The correlation between technology and the growth of human population is deceptively misleading and it leads quickly to a dubious loop (Korotayev, Malkov, & Khaltourina, 2006a): technology increases the carrying capacity, the increased carrying capacity promotes population growth, population growth promotes the growth of technology, technology increases the carrying capacity, and so on. It is explaining one unknown mechanism by another unknown mechanism. It is going in circles and explaining nothing.

Technology might be helpful in supporting the existence of people but is it essential? When trying to explain the mechanism of growth it is necessary to consider first the most obvious and most essential force or forces. Other forces may be added if the essential force is insufficient to explain growth.

It is obvious that the essential force controlling the growth of population is the force of procreation. Technology does not produce children and it is even not essential to support growth, as demonstrated by the fast growth of population in poor countries.

Even if we briefly agree that technology supports or limits the growth of human population, such an "explanation" ignores the obvious and *indispensable* force of growth of human population, the force of procreation. It ignores the abundant evidence that even without advanced technology people can still produce children and support them.

This closed-loop explanation is supported by the assertion that "throughout most of human history the world population was limited by the technologically determined *ceiling* of the carrying capacity of land" (Korotayev, Malkov, & Khaltourina, 2006a, p. 18. Italics added.). It is a typical claim based of pure imagination, a statement that has to be accepted by faith. How can we possibly prove that over thousands of years and all over the world, the growth of human population was so *finely tuned* to the "the technologically determined ceiling of the carrying capacity of land"?

When this statement was published and when the associated closed-loop explanation was proposed it was already well known that the growth of human population was hyperbolic, at least during the AD era (Kapitza,1992, 1996, 2006; Kremer, 1993; Podlazov, 2002; Shklovskii, 1962, 2002; von Foerster, Mora, & Amiot, 1960; von Hoerner, 1975). Evidence-based indication is that hyperbolic growth was in general unconstrained and surprisingly robust over a long time. This type of growth contradicts the concept of the limiting effects of the ceiling of the carrying capacity. This ceiling appears to have been always much higher than required for supporting growth, the conclusion being in agreement with the study of the ecological capacity and ecological footprints showing that only recently we have crossed the ecological limit of our planet (Ewing, *et al*. 2010).

To accept this closed-loop explanation we would have to accept, without a proof, that each component in this loop was not only finely tuned but also that they were all for some mysterious and unexplained reason increasing hyperbolically: the population was increasing hyperbolically, the technology was increasing hyperbolically, the carrying capacity was increasing hyperbolically and all of them were so finely tuned as to increase in unison, in such perfect harmony and so close to each other. The size of the population would have to be all the time close to the limiting ceiling of the carrying capacity, which would be so mysteriously increasing.

The proposed closed-loop explanation breaks also down already in the first step. What if the carrying capacity was already so large that the assumed contribution from technology was inconsequential? The size of the population in the past was small over a long time. It is hard to accept that our planet was incapable to support the increasing population.

With the exception of just two demographic transitions in the past 12,000 years (Nielsen, 2016a), the growth of human population was increasing without any major disturbance. With the small number of people and with the huge resources of our planet we can reasonably expect that the carrying capacity was much higher than the size of human population.

It would be unrealistic and unconvincing to assume that the growth of human population over such a long time was so precisely adjusted to the carrying capacity. It would be unrealistic to expect that this *fine tuning* was done so precisely by technological development. To make such a claim we would first have to prove that the growth of human population was always limited by the carrying capacity of our planet but we have no such proof and probably we shall never have. Any theory, which attempts to explain the mechanism of growth of human population by *fine tuning* of the size of population to the carrying capacity by technology, economic growth or by any other means is either unscientific (because it is based on untestable assumptions) or at least strongly questionable.

We would have to have some incredibly advanced technology to *measure* the carrying capacity and to *adjust* the growth of human population so precisely to its "ceiling." But even then, we could hardly expect such a regular hyperbolic growth. By using this advanced technology, we would also have to control precisely three interacting processes: technological development, the increase in the carrying capacity and the growth of human population. We would have to make sure that these three processes are perfectly synchronised and that they follow the closely coupled hyperbolic trajectories.

To justify the closed-loop process we would have to explain it without assuming that it was controlled by any advanced technology. Without such explanation, the mechanism of the proposed closed-loop remains unexplained and consequently, it does not explain the mechanism of the growth of human population.

We can also have other questions about this first step in the postulated closed loop. What is the carrying capacity of our planet? What was the carrying capacity of our planet over the past 12,000 years or longer? What was the contribution of technology to the carrying capacity? Even if we assume that technology increases the carrying capacity, is this assumed increase so essential to support the growth of human population? It is well known that people can survive on very little and that even then they can still procreate and support children. All they need is basic food, body cover and shelter.

How much damage is caused by technology? How is the technology *reducing* the carrying capacity? Can we ignore, for instance, that carbon footprint accounts for about 50% of our total ecological footprint? (Ewing, *et al*. 2010). Can we ignore the pollution of not only the atmosphere but also of the land and water? Can we ignore climate change, the ever-increasing weather-related economic losses, the decreasing carrying capacity of people living on islands, the increasing deforestation, the continuing human-induced extinction of species, the continuing loss of arable land, the overuse of pesticides, herbicides, artificial fertilisers and other agricultural chemicals? Can we ignore the ever-increasing urban population and their increasing dependence on food supply, which comes from the decreasing land resources? Can we ignore how the huge and the well-stocked arsenal of weapons is relentlessly used to destroy the carrying capacity? Can we ignore the never-decreasing stream of displaced population?

If we want to claim that technology increases the carrying capacity, we should also consider how this carrying capacity is decreased by technology. But the essential point is to show that technology was indeed playing the crucial role in shaping the growth of population, that this assumed force of growth has to be added to the essential and indispensable force of procreation, that without technology population would not have been increasing or that it would not have been increasing hyperbolically.

Another problem with linking technological development with the growth of human population is the misinterpretation of the fundamental mechanism of technological development. Technological growth is not prompted by the sheer number of people but by concepts, ideas and solutions. This is the driving force of technological development. People are just *carriers* of these concepts, ideas and solutions, or more precisely, carriers of the genetic ability to generate concepts, ideas and solutions.

Is the technological development dependent on the *number* of people? While it is true that with a larger number of people we can expect a greater number of ideas and solution, it is also true that the growth of human population is now slowing down. Does it mean that technological development is also slowing down because of the slowing down growth of human population? If the growth of human population is going to reach a maximum and stop growing, will the technology also reach a certain maximum and stop growing? Will people stop thinking and inventing?

The growth of technology is not determined by the number of people but by the number of creative ideas, inventions and solutions, which do not appear to be directly proportional to the number of people. Consequently, even if the size of population is going to be constant, people will not stop being intellectually active.

The correlation between technology and the growth of human population was investigated by Kremer (1963). He claims that there is a close correlation between the growth of population and technological development, which is hardly surprising. However, by observing a correlation between two processes we can only tell that there is a correlation. The correlation alone does not explain the mechanism of growth of any of the correlated processes. Correlations can be strongly misleading and they have to be handled with care.

Kremer claims that the growth rate of human population during the AD era was approximately proportional to the size of human population indicating that the growth was hyperbolic but he did not explain *why* it was hyperbolic. He suggests the correlation between the growth of human population and the growth of technology but this correlation does not explain the mechanism of growth of any of them. It does not explain why these two correlated processes are hyperbolic. It is like with the finely-tuned closed-loop mechanism proposed by Korotayev, Malkov & Khaltourina, (2006a): one process is explained by another without explaining any of them. The growth of human population is hyperbolic because the growth of technology is hyperbolic, and the growth of technology is hyperbolic because the growth of population is hyperbolic. It is also explaining one unknown mechanism by another unknown mechanism and going in circles.

The primary, if not the only, force driving the growth of population is the force of procreation, which in its simplest representation is the biologically controlled sex drive and biologically controlled mortality. Until recently, children were not produced by technology. Mortality was also not controlled by technology. Maybe technology could be claimed to give a better chance of survival but it is definitely not the primary force of growth. Likewise, the primary force driving technological development can be identified as concepts and ideas created by people, combined with the efficiency of sharing information.

The primary force of the growth of population is represented by the biological processes controlling birth and death. The primary force controlling the growth of technology is represented by concepts, ideas and generally by creative activities of human population. Biological process controlling birth and death apply not only to humans but also to other species. The force of creative thinking applies specifically only to humans. There might be some examples of creative thinking in other species, particularly in primates, but they are on such a low level that they do not initiate some new lines of technological development.

If the force responsible for the growth of technology were determined by the sheer number of people, i.e. by the number of members of this particular species, there would be no reason for excluding other species from this process. They should be also expected to develop technology but they do not. The growth of technology and the growth of population are controlled by different forces of growth. Their fundamental mechanisms of growth are distinctly different.

There is a close correlation between the growth of technology and the growth of human population only because creative concepts come from humans. Explaining the growth of population by technology and technology by the growth of population is going in circles and explaining nothing.

3. Convoluted construction

Hyperbolic growth is described by a simple formula:

$$S(t) = \frac{1}{C - kt}, \tag{1}$$

where $S(t)$ is the size of the growing entity, such as population or the Gross Domestic Product (GDP), C is the constant of integration, k is a positive constant and t is time.

This expression is a solution of a very simple differential equation:

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = kS(t). \tag{2}$$

Normally, the next step would be to explain why the growth is hyperbolic. To this end, we would have to start with some simple and easily acceptable assumptions and *derive* the hyperbolic formula based on these assumptions. Maybe we could also start with acceptable assumptions and derive and alternative formula, which would give a better description of data. However, if we derived a more complicated formula, which would not give a better description of data we could then decide that we were on the wrong track and we would have to try another approach.

In contrast, in the demographic and economic research there appears to be a tendency to *construct* mathematical formulae and to try to make them as complicated as possible. Here is one such example (Johansen & Sornette, 2001).

Start with the logistic equation of growth

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = b[K - S(t)]. \tag{3}$$

This is already a questionable starting point because we know that population and the GDP do not grow logistically but hyperbolically. Even now, they do not yet level off (Nielsen, 2016d) to suggest a conversion to a logistic-type of growth.

Assume that the limit to growth K depends on time.

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = b[K(t) - S(t)]. \tag{4}$$

For no apparent reason, delete $S(t)$ from the right-hand side of the eqn (4).

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = b[K(t)]. \tag{5}$$

Again, for no apparently good reason, assume that

$$K(t) = [S(t)]^{\delta}, \tag{6}$$

where $\delta > 1$.

Under this assumption, eqn (6) is now changed to

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = b[S(t)]^{\delta}. \tag{7}$$

This equation can be presented as

$$\frac{dS(t)}{dt} = b[S(t)]^{\delta+1}. \tag{8}$$

We can solve it by substitution $S = Z^{-1}$. The solution is

$$\frac{1}{\delta}S^{-\delta} = c - bt \qquad , \tag{9}$$

where *c* is the constant of integration. So now we have

$$S(t) = (b\delta)^{z}(t_c - t)^{z}, \tag{10}$$

where $z = -1/\delta$ and $t_c = c/b$ is the time of singularity when $S(t)$ escapes to infinity.

Replace $(b\delta)^{z}$ by an arbitrary and adjustable parameter *B* and add another arbitrary and adjustable parameter *A* to construct

$$S(t) = A + B(t_c - t)^{z}. \tag{11}$$

Assume that the parameter *z* is a complex number

$$z = -(\beta + i\omega) \tag{12}$$

So now we have

$$S(t) = A + B(t_c - t)^{\beta+i\omega}. \tag{13}$$

Find the real component of $(t_c - t)^{\beta + i\omega}$.

This is an easy exercise that can be completed using two well-known formulae:

$$x^y = e^{y \ln x} \tag{14}$$

and

$$e^{i\varphi} = \cos\varphi + i \sin\varphi . \tag{15}$$

The answer is

$$\mathrm{Re}(t_c - t)^{\beta + i\omega} = (t_c - t)^{\beta} \cos[\omega \ln(t_c - t)]. \tag{16}$$

Assuming that both *A* and *B* are real, the formula for $S(t)$ can be now expressed as

$$\mathrm{Re}\, S(t) = A + B\, \mathrm{Re}(t_c - t)^{\beta + i\omega}, \tag{17}$$

which with the help of the eqn (16) gives

$$\mathrm{Re}\, S(t) = A + B(t_c - t)^{\beta} \cos[\omega \ln(t_c - t)]. \tag{18}$$

Use the eqn (13) again but now delete $i\omega$.

$$S(t) = A + B(t_c - t)^{\beta} . \tag{19}$$

Return to the eqn (16) and multiply the right-hand side of this equation by a constant *D*.

$$\mathrm{Re}(t_c - t)^{\beta + i\omega} = D(t_c - t)^{\beta} \cos[\omega \ln(t_c - t)]. \tag{20}$$

Add a phase shift in the eqn (20).

$$\mathrm{Re}(t_c - t)^{\beta + i\omega} = D(t_c - t)^{\beta} \cos[\omega \ln(t_c - t) + \phi]. \tag{21}$$

Return to the equation (19) and add to it the right-hand side of the eqn (21). We have now *constructed* the equation published by Johansen & Sornette (2001).

$$S(t) = A + B(t_c - t)^{\beta} + D(t_c - t)^{\beta} \cos[\omega \ln(t_c - t) + \phi] \tag{22}$$

This equation contains seven adjustable parameters but we do not know how they are supposed to be linked with the mechanism of growth. We know how we *constructed* (not derived) this complicated and impressive formula but we do not know why we have it and indeed why we should be interested in using it except perhaps to draw a line through data points, which we could do equally successfully using pen and paper and obtain equally meaningless result.

It is always good to look for mathematical description of data because it could help in understanding the nature of the observed phenomenon. However, if complicated description is not better than description given by a simple mathematical formula there is obviously no advantage in using the complicated description.

Hyperbolic growth described by the eqn (1) gives a satisfactory description of the growth of population and of the economic growth (Nielsen, 2014, 2016a, 2016b, 2016c). This is a simple formula, which could be expected to have a simple explanation. But now, we have a significantly more complicated formula. So, rather than making our task of explaining the mechanism of growth easier we have made it even more complicated.

In Figure 1, the distribution generated by the complicated eqn (22) is compared with the first-order hyperbolic distribution described by the eqn (1) and with data. As explained elsewhere (Nielsen, 2016a), fitting data around AD 1 by using hyperbolic distribution is pointless because around that time there was a transition from a fast to a slow hyperbolic trajectory. However, if we replace the complicated formula of Johansen and Sornette by a significantly simpler reciprocal of the second order polynomial

$$S(t) = (a_0 + a_1 t + a_2 t^2)^{-1} \tag{23}$$

we can generate a virtually identical distribution. There is no clear advantage in using the complicated formula of Johansen and Sornette. Simple description using the first-order hyperbolic distribution given by the eqn (1) gives acceptable representation of data but we can also replicate the complicated seven-parameter calculations but using just three parameters.

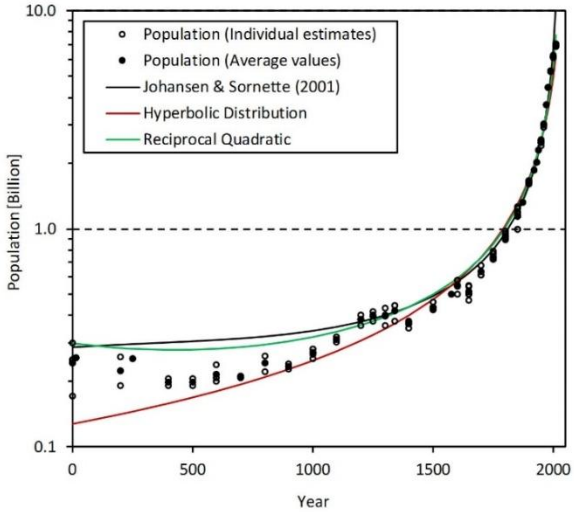

Figure 1. *Growth of the world population calculated using the Johansen & Sornette's (2001) constructed formula (22) is compared with the calculations based on significantly simpler formulae given by the eqns (1) and (24). Population data come from numerous sources compiled by Manning (2008) and by the US Census Bureau (2016). The parameters for the distribution of Johansen and Sornette given by the eqn (22) are: A≈0, B≈1624,D ≈−127, z≈−1:4, t_c≈2056; ω≈6:3 and ϕ≈5.1. Parameters for the hyperbolic distribution given by the eqn (1) are:* $C = 7.875\times10^0$ *and* $k = 3.834\times10^{-3}$. *Parameters for the reciprocal second-order polynomial distribution given by the eqn (24) are:* $a_0 = 3.367\times10^0$, $a_1 = 1.172\times10^{-3}$ *and* $a_2 = -1.382\times10^{-6}$.

The aim of constructing this complicated formula appears to be misplaced because even Johansen and Sornette used a significantly simpler formula in their analysis of a wide range of data presented in their Figs 9-32. The formula they used was

$$S(t) = a(t_c - t)^z . \tag{24}$$

However, even in this simplified form it is already unnecessarily more complicated than the eqn (1) because $S(t)$ is no longer represented by the reciprocal of a linear function but by the time difference taken to the power of *z*. This expression is linear only if $z = 1$. For integer values of $z > 1$ it describes higher-order polynomials. For integers $z < -1$ it describes reciprocals of higher order polynomials. However, *z* can be also any other number greater or smaller than zero.

4. The homeostatic simulation model

In conformity with the generally accepted established knowledge in demography and in economic research (Nielsen, 2016d), Artzrouni & Komlos (1985) imagined that the growth of population can be divided into two distinctly different regimes: Malthusian stagnation and explosion. These regimes of growth are assumed by be controlled by two distinctly different mechanisms of growth. They assumed incorrectly that the growth before the Industrial Revolution was controlled by random forces such as wars, famines and diseases, the mechanism causing allegedly stagnation in the growth of population. They also assume, incorrectly, that the growth after around the Industrial Revolution was exponential.

They should have known that their assumptions were unrealistic and incorrect because many years earlier it has been shown that the growth of population was hyperbolic (von Foerster, Mora, & Amiot, 1960; von Hoerner, 1975). Hyperbolic growth cannot be divided into two regimes of growth, slow and fast. For this type of growth, Malthusian regime does not exist and the apparent explosion is just the natural continuation of hyperbolic growth. There was no stagnation in the growth of human population and in the economic growth and there were no takeoffs leading to distinctly different explosive growth (Nielsen, 2014, 2015, 2016a, 2016b, 2016c, 2016e, 2016f, 2016g, 2016h).

Their work is important because, unknown to them, they have demonstrated that the established knowledge is contradicted by science. They did not realise that they made an important discovery because typically for the research carried out within the constraints of the established knowledge they did not compare results of their research with data.

To generate the growth of population before the Industrial Revolution, Artzrouni and Komlos carried out Monte Carlo simulations of the alleged Malthusian regime of stagnation. To describe the alleged population explosion, they simply assumed exponential growth after the Industrial Revolution. In their model, the growth of population is given by

$$\frac{\Delta S(t)}{\Delta t} = rS(t) . \tag{25}$$

For the constant *r*, this equation would describe exponential growth. However, in their calculations, the growth rate *r* is either constant (after the Industrial Revolution) or time-dependent (before the Industrial Revolution).

So, more explicitly, they consider two stages of growth. Before the Industrial Revolution the growth is given by:

$$\frac{\Delta S(t)}{\Delta t} = r(t)S(t), \tag{26}$$

whereas after the Industrial Revolution it is given by

$$\frac{\Delta S(t)}{\Delta t} = r_e S(t), \tag{27}$$

where r_e is a certain constant "escape rate" (Artzrouni & Komlos, 1985, p 27), escape from nowhere because there was no escape, or more accurately there was nothing to escape from, because the mythical Malthusian trap did not exist. The growth of population was monotonically hyperbolic, and the Industrial Revolution had no impact on changing the growth trajectory. However, according to the established but erroneous knowledge, there was an escape.

Fluctuations in the growth rate $r(t)$ before the Industrial Revolution are determined by $e(t)$ described as "a non negative random variable generated by a Monte Carlo type of simulation" (Artzrouni & Komlos, 1985, p. 27). For no apparent reason, this variable is defined by the following equation:

$$e(t) = 0.1v(t)U(t)\{1 + e^{0.15[y(t)-5]}\}, \tag{28}$$

where $v(t)$ is a random number drawn from a normal distribution with the mean 0 and variance 1, $y(t)$ is the number of decades the population was in the assumed Malthusian crisis and $U(t)$ is defined (again for no clear reason) as

$$U(t) = \frac{1}{1 + 4e^{-400[P_T \cdot P(t)]}}. \tag{29}$$

The population is divided into two sectors: the subsistence sector ("rural") and the capital producing sector ("urban"). In the eqn (29), $P(t)$ represents the per capita output (production) of the subsistence sector. If the per capita output is below a certain threshold defined by P_T, i.e. if $P(t) < P_T$, the population is assumed to be in the Malthusian crisis and cannot grow. If $P(t) \geq P_T$, the population is assumed to be out of crisis and can increase.

The per capita output in the subsistence sector is defined as

$$P(t) = C_2[K(t)]^{\beta_1} \frac{[S_R(t)]^{\beta_2}}{S(t)}, \tag{30}$$

where C_2, β_1 and β_2 are positive constants with $\beta_1 + \beta_2 = 1$, $K(t)$ is the capital stock and $S_R(t)$ is the population in the subsistence ("rural") sector.

The total output in the subsistence sector is given by

$$Q_R(t) = C_2[K(t)]^{\beta_1}[S_R(t)]^{\beta_2}. \tag{31}$$

Likewise, the total production in the capital producing sector ("urban") is given by

$$Q_U(t) = C_1[K(t)]^{\alpha_1}[S_U(t)]^{\alpha_2}, \tag{32}$$

where C_1, α_1 and α_2 are positive constants with $\alpha_1 + \alpha_2 = 1$.
The total population is then given by

$$S(t) = S_R(t) + S_U(t). \tag{33}$$

Returning to the eqn (28) we should notice that the function $U(t)$ defining the time-dependent parameter $e(t)$, which plays the essential role in the Monte Carlo simulations, depends on $P(t)$, which in turn depends on the capital stock $K(t)$. The growth of the capital stock is described as

$$\frac{\Delta K(t)}{\Delta t} = \lambda(t) Q_U(t), \tag{34}$$

where $\lambda(t)$ is defined as

$$\lambda(t) = 0.01 + 1.778 \cdot 10^{-26} e^{0.05756t}. \tag{35}$$

The process of Monte Carlo simulations is well described by Artzrouni & Komlos (1985). These simulations produced most interesting results. Designed to demonstrate the existence of Malthusian regime of stagnation, *the model shows that the regime of Malthusian stagnation did not exist*. The assumed mechanism of stagnation does not produce stagnation but a steadily-increasing growth. Furthermore, the growth generated by the model during this imaginary regime of stagnation does not fit the data (see Figure 2).

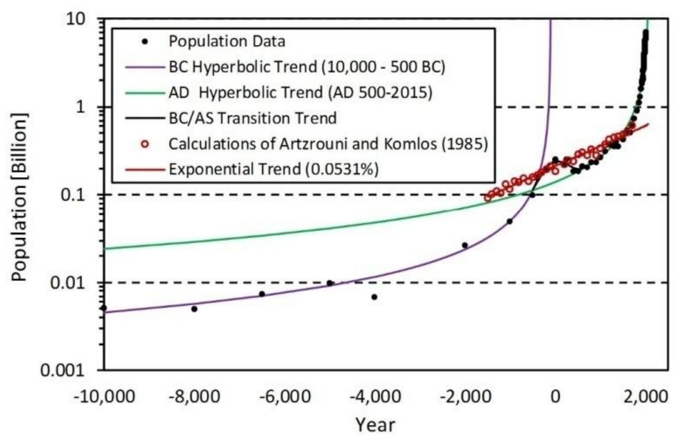

Figure 2. *The established knowledge in demography is contradicted by science. Simulations of the mechanism of Malthusian stagnation carried out by Artzrouni & Komlos (1985) do not produce stagnation but a steadily increasing exponential growth. They also do not fit date. These calculations are compared with hyperbolic distributions (Nielsen, 2016a). The data represent the average values of the size of population calculated using the compilations of Manning (2008) and of the US Census Bureau (2016).*

Parameters describing hyperbolic distributions shown in Figure 2 and defined in the eqn (1) are: $C = -2.282\times10^{0}$ and $k = 2.210\times10^{-2}$ for the BC era and $C = 7.061\times10^{0}$ and $k = 3.398\times10^{-3}$. The data come from a variety of sources compiled by Manning (2008) and the US Census Bureau (2016).

The model of Artzrouni & Komlos's (1985), designed to reproduce the "well documented fluctuations experienced by the world's population throughout history" (Artzrouni & Komlos, 1985, p. 24), produced instead a steadily increasing growth along exponential trajectory. (In the semilogarithmic display, exponential growth is represented by an increasing straight line.) Furthermore, their model calculations do not fit the data. Their results show clearly that the model of Malthusian stagnation does not work. The mechanism of Malthusian stagnation does not describe the growth of population. The model based on the assumption of the mechanism of Malthusian stagnation did not generate the required fluctuations in the growth of population let alone fluctuations that "were, to a large extend, brought about by randomly determined demographic crises (wars, famines, epidemics, etc.)" (Artzrouni & Komlos, 1985, p. 24).

Thus, if Artzrouni and Komlos took the final step normally expected in scientific investigations, if they compared theory with data, even with the data used by von Foerster, Mora & Amiot (1960), they would have made an important discovery that the fundamental concepts of the established knowledge in demography are incorrect. Theywould then be able to suggest new lines of research.

It is essential to notice that even though Monte Carlo simulations based on the assumption of the mechanism of Malthusian stagnation produced exponential growth it would be incorrect to claim that the mechanism of Malthusian stagnation generates exponential growth. Equation (26) makes it clear that Artzrouni & Komlos (1985) *assumed* exponential growth. They *assumed* that Monte Carlo calculations were fluctuating around the growth rate describing exponential growth because eqn (26) describes modulated exponential growth. If we assume exponential growth it is hardly surprising that we get exponential growth. Fluctuations in the growth rate are not readily reflected as fluctuations of the growth of population or the GDP (Nielsen, 2016i, 2016k).

5. Lagerlöf's model of growth

Lagerlöf's model of growth (2003a, 2003b) belongs to the so-called OLG (overlapping generations) models (Aliprantis, Brown & Burkinshaw, 1990) used for instance by Becker, Murphy & Tamura (1990) and by Galor (2005a, 2011) to look at the growth of the population from the economic perspective. The central idea of this approach is to try to explain the growth of population by considering human capital defined as "embodied knowledge and skills" (Becker, Murphy & Tamura, 1990, p. S13). The growth is on the favourable rates of return.

> When human capital is abundant, rates of return on human capital investments are high relative to return on children, whereas when human capital is scarce, rates of return on human capital are low relative to those on children. As a result, societies with limited human capital choose large families and invest little in each member; those with abundant human capital do the opposite (Becker, Murphy & Tamura, 1990, p. S35).

It is a strong assumption, which is hard to accept. One would have to have a strong proof that this assumption is correct but we do not have such a proof.

It is interesting that neither Becker, Murphy & Tamura (1990), nor Galor (2005a, 2011), nor Lagerlöf (2003b) tried to compare their model predictions with population data. Lagerlöf (2003b) came close to testing his model against data when he generated growth rates in his Monte Carlo simulations but we shall show that his model is in disagreement with data he was referring to in his publication.

Lagerlöf's model is an excellent example of convoluted models characterised by the abundance of parameters but models, which neither describe data nor explain the mechanism of growth. This model was also designed to reproduce the epoch of stagnation and the alleged transition from stagnation to growth at the time of the Industrial Revolution, all as specified by the prescribed instructions of the established knowledge in demography and in the economic research. Like Artzrouni & Komlos (1985), Lagerlöf was also on the verge of making an important discovery that the established knowledge in demography and in economic research is contradicted by science. Like Artzrouni & Komlos (1985), he was on the verge of proving that the epoch of Malthusian stagnation did not exist and that there was no transition from stagnation to growth. Like Artzrouni and Komlos (1985), he was on the verge of showing that simulations of Malthusian stagnation do not produce stagnation, that they do not fit data and that they do not explain the mechanism of growth. He missed making this important discovery because he did not take the final step normally expected in scientific investigations – he did not compare theory with data. Parameters and definitions used in Lagerlöf's model are listed in Table 1.

Table 1. *Parameters and definitions used in Lagerlöf's theory (Lagerlöf, 2003a, 2003b)*

Parameter	Description
t	Time interval or "period t" assumed to be 25 years, i.e. one generation
H_t	Human capital or "a component resulting from parental investment" (Lagerlöf, 2003b, p. 426) called also "human capital stock" (Lagerlöf, 2003a, p. 760)
L	The "units of skills" (Lagerlöf, 2003b, p. 426) endowed by nature to "every agent" (person)
$L + H_t$	The "productivity of a unit of time" (Lagerlöf, 2003b, p. 426)
v	"a fixed time cost of rearing one child" i.e. "the time required to nurse the child just enough to keep her alive" (Lagerlöf, 2003a, p. 759)
ρv	Assuming $0 < \rho < 1$, this product "measures the direct inheritance of human capital from one generation to the next" reflecting the assumption that less than 100% of the time invested in rearing (nursing) a child is converted into human capital.
h_t	The time invested in the education of each child
l_t	The "time input in the consumption good sector" (Lagerlöf, 2003a, p. 759) i.e. time spent on production or work
ω_t	The "mortality shocks" (Lagerlöf, 2003b, p. 426) "which can be interpreted as epidemics" (Lagerlöf, 2003a, p. 760), the function assumed to be described by the probability density function of a log-normal distribution.
P_t	"the (adult) population size" called also "population density" in the generation t (Lagerlöf, 2003a, p. 760). The fundamental assumption of OLG models is that people live only for two generations. All adults in the generation t are replaced by the children born during the generation t. This new generation will be completely replaced by the next generation.
$A(P_t)$	The productivity parameter, which enters into the equation of the time-dependence of human capital
B_t	"the number of born children (or births)" (Lagerlöf, 2003a, p. 759). It is the average number of children per capita of adult population born in the generation t, i.e. over the entire 25 years.
T_t	The "survival rate" (Lagerlöf, 2003a, p. 760). It is the average *fraction* of the number of individuals born during the 25 years of the generation t, who survive to the next generation $t + 1$.
$(v + h_t)B_t$	The total time invested in children per capita of the adult population calculated over the entire time of one generation, i.e. over the total time of 25 years
Y_t	The output of the consumption (production) of goods
C_t	The "adult consumption" (production) (Lagerlöf, 2003b, p. 426)
α	A parameter ($\alpha > 0$) used in the utility function

Assuming that each person (agent) is endowed with a unit of time, the time budget for each agent is given by

$$1 = l_t + (v + h_t)B_t \tag{36}$$

At any given time, each person (agent) is assumed either to work or to spend time with children.

Assuming a single economy (or non-interacting economies) and that children consume (produce) nothing, the output of the consumption (production) of goods is given by

$$Y_t = l_t(L + H_t) = C_t \tag{37}$$

It is simply the productivity per unit of time multiplied by the time spent at work.

The survival rate is given by

$$T_t = \frac{H_t}{S_t} \frac{1}{\omega_t + H_t / S_t} \tag{38}$$

In the absence of mortality shocks ($\omega_t = 0$), the survival rate $T_t = 1$.

The production of human capital is given by

$$H_{t+1} = A(P_t)[L + H_t](\rho v + h_t) \tag{39}$$

Human capital increases in proportion to the productivity per unit of time multiplied by the time spent with each child, with the part of this time corrected for the unproductive fraction of time when nursing a child. By including the parameter $0 < \rho < 1$ it is assumed that education is more profitable for the increasing of human capital than nursing.

Each agent is assumed to maximise a utility function describing personal preferences and is given by

$$U_t = \ln(C_t) + \alpha \ln(B_t T_t) + \alpha\delta \ln(L + H_{t+1}) \tag{40}$$

The first term of the utility function measures the utility (the preference) of consumption (production), the second measures the utility of surviving children given by $B_t T_t$ and the third the utility of human capital of the offspring.

By maximising the utility function, we get the following expression for the optimal (preferred) number of births

$$B_t = \left(\frac{\alpha}{1 + \alpha}\right) \frac{1}{v + h_t} \tag{41}$$

The number of born children depends entirely on the time invested in each child corrected by a factor dependent on the parameter used in the utility function. The larger the invested time, the smaller is the number of children, or vice versa.

The annual crude birth and death rates ($B_{r,t}$ and $D_{r,t}$, respectively) are calculated using the following expressions:

$$B_{r,t} = 1000\left(B_t^{1/25} - 1\right) \tag{42}$$

$$D_{r,t} = 1000\left(1 - T_t^{1/25}\right) \tag{43}$$

Calculations become significantly more complicated if interacting countries are included. Thus, for instance, assuming that that a demographic shock in one country is also reflected in other countries, the survival rate can be expressed as

$$T_{i,t} = \frac{H_{i,t} + \sum_{j \neq i}^{N-1} k_{ij} H_{j,t}}{\omega_{i,t} P_{i,t} + \sum_{j \neq i}^{N-1} k_{ij} \omega_{j,t} P_{j,t} + \left[H_{i,t} + \sum_{j \neq i}^{N-1} k_{ij} H_{j,t} \right]} \tag{44}$$

If we look back at eqns (38), (42) and (43) we can see that when $\omega_t = 0$, then $T_t = 1$ and the death rate $D_{r,t} = 0$, which also means that if mortality shocks are low, i.e. if $\omega_t \approx 0$, the death rate is also approximately zero. If we assume that the time spent with each child remains approximately the same over time, or equivalently that the number of born children remains approximately the same, a dramatic decrease in mortality shocks should generate a prominent population explosion.

This mechanism is the essence of the Demographic Transition Theory, which claims that towards the end of the assumed first stage of human history, interpreted as the epoch of Malthusian stagnation, the death rate started to fall while the birth rate remained approximately the same, the process creating allegedly population explosion, the explosion which in fact never happened. This is also the essence of the three regimes of growth postulated by Galor & Weil (1999, 2000) but contradicted by the analysis of data (Nielsen, 2016f).

The mechanism of Malthusian stagnation followed by explosion is carefully incorporated in the Lagerlöf's model of growth. In particular, regarding the mortality shock function ω_t, Lagerlöf explains:

> To understand the mechanisms driving the results in the calibration later, it is useful to first think of economies where ω_t is constant over time: Either high or low. To replicate the Three Regimes of Galor & Weil (1999, 2000), discussed in the introduction, we shall rig the model so that a high-ω economy converges toward a locally stable (Malthusian) steady state, whereas a low-ω economy converges to a balanced growth path (Lagerlöf, 2003a, 763).

The three regimes Lagerlöf is writing about are the assumed Malthusian regime of stagnation, which was supposed to last for thousands of years but which never existed; the post-Malthusian regime marked allegedly by the rapid increase of population and economy; and the modern growth regime or sustained growth regime, which allegedly follows a little later but which also represents an imaginary stage of growth. We have already demonstrated that these three regimes of growth did not exist (Nielsen, 2016f).

Lagerlöf's theory is based on the scientifically contradicted fantasy and if he carried his research properly, if he compared his theory with empirical evidence, he would have soon discovered that he was guided by fiction. His hard and convoluted work was unnecessary because it has been known for a long time that the growth of population was hyperbolic (Kapitza,1992, 1996, 2006; Kremer, 1993; Podlazov, 2002; Shklovskii, 1962, 2002; von Foerster, Mora, & Amiot, 1960; von Hoerner, 1975). Hyperbolic growth can be described by an exceptionally simple mathematical formula, which is just the reciprocal of a linear distribution. This type of growth is in contradiction of the concepts of stagnation and explosion.

Using his model and Monte Carlo simulations, Lagerlöf generated growth rate for the growth of population in England, France and Sweden (Lagerlöf, 2003b). His model produced *minor* fluctuations in the growth rate, which were interpreted by Lagerlöf as the proof of the existence of the regime of Malthusian stagnation. That was a serious mistake because even large fluctuations in the growth rate are not readily reflected in the growth of population (Nielsen, 2016i, 2016j), and we do not even have to carry out laborious calculations to see that fluctuations in the growth rate are not reflected as similar fluctuations in the growth of population. Data for Sweden are well known (Statistics Sweden, 1999). They are often used in defence of the Demographic Transition Theory without even realising that the they are in its contradiction. There, in the same document, for everyone to see, we have *graphs* showing fluctuating birth and death rates, and fluctuating annual population increase but also we have a graph of population growth with no signs of fluctuations. The usual practice of showing fluctuations in birth and death rates or in the growth rate and claiming that we have a proof of the existence of Malthusian stagnation is unjustified. These fluctuations are not reflected in the growth of population and consequently they have no impact on the mechanism of growth. They are, in this respect, irrelevant.

Figure 3 shows an example Lagerlöf's results for France. His model-generated growth of population was calculated using the numerical integration of the following differential equation:

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = R_L(t), \tag{45}$$

where $R_L(t)$ is the Lagerlöf's, model-generated and*fluctuating* growth rate, precisely as published in his paper (Lagerlöf, 2003b).

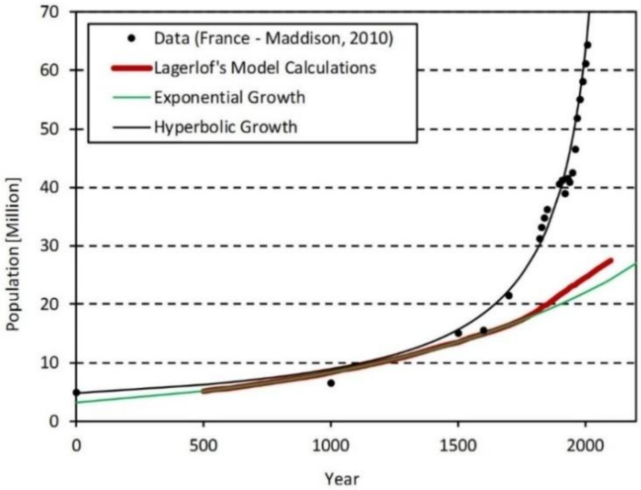

Figure 3. *The established knowledge in demography is contradicted by science. Simulations of Malthusian stagnation carried out by Lagerlöf (2003b) do not produce stagnation but a steadily increasing growth of population. Furthermore, his model calculations do not fit data (Maddison, 2010). Model-calculated distribution follows exponential trajectory because the growth rate was* assumed *to oscillate around a constant value. The claimed population explosion is just a small deviation from the exponential trajectory at its end. The growth of population in France was hyperbolic.*

In Figure 3, Lagerlöf's model-generated distribution is compared with the exponential distribution and with data. We also show the hyperbolic distribution fitting the data (Maddison, 2010). These data were not available to Lagerlöf but he had access to similar data (Maddison, 2001) published before the publication of his work.

Parameters describing hyperbolic distribution are: $C = 2.085 \times 10^{-1}$ and $k = 9.635 \times 10^{-5}$ [see eqn (1)]. The exponential distribution, which is so closely followed by Lagerlöf's model-generated results, is described by the following equation:

$$S(t) = C' e^{rt} . \quad (46)$$

Its parameters are $C' = 3.100 \times 10^{0}$ and $r = 9.780 \times 10^{-4}$.

The tiny, model-generated fluctuations in the growth rate presented in Lagerlöf's publication (Lagerlöf, 2003b) could not have possibly generated oscillations in the growth of population. Even large fluctuations are not readily reflected in distributions describing growth, such as growth of population or the GDP (Nielsen, 2016i, 2016j). Lagerlöf could have seen it clearly if he looked at the data for Sweden (Statistics Sweden, 1999). He could have also known it if he studied the excellent data for England (Wrigley & Schofield, 1981). These results show clearly, even without carrying out any calculations, that even large fluctuations in birth and death rates and in the corresponding growth rate have no tangible effect on the growth of population and consequently that they have no effect on shaping the mechanism of growth. These data are clearly contradicting the established knowledge in demography but they are systematically ignored. The established knowledge in demography is also contradicted by results published over 50 years ago (von Foerster, Mora, & Amiot, 1960) but they are also systematically ignored.

The important contribution of Lagerlöf's Monte Carlo simulations is to show that the mechanism of Malthusian stagnation and population explosion does not work. Such a mechanism fails to produce the desired effects of stagnation and explosion and it fails to fit the data.

Results of Lagerlöf show that his model-generated distribution was exponential and that his claimed population explosion is just a minor deviation from the exponential trajectory. However, one might wonder why model-generated results follow exponential trajectory. Does it mean that the process of Malthusian stagnation generates exponential growth? No, it does not. Results depend on our assumptions about the way birth and death rates are fluctuating.

Lagerlöf *assumed* that crude birth rate was a non-zero constant and that crude death rate fluctuated around a non-zero *constant* value. Naturally, therefore, his growth rate also fluctuated around a non-zero constant, which in turn generated exponential growth. If Lagerlöf assumed that birth rate was zero and that the death rate fluctuated also around zero, he would have produced growth rate fluctuating around zero and thus he would have produced a constant size of population in his Monte Carlo calculations but he would still have not produced the required fluctuations in the size of population and his results would have been in a clear disagreement with data. The same applies to the calculations of Artzrouni & Komlos (1985). If they did not assume the modulated exponential growth during the postulated epoch of Malthusian stagnation [see the eqn (26)], they would have also produced a constant population without the so-called Malthusian oscillations.

If Lagerlöf took the final step and compared his model-generated distributions with data (Maddison, 2001), if he consulted the available to him literature (Statistics Sweden, 1999; Wrigley & Schofield, 1981) he would have made an important discovery that the concept of Malthusian stagnation followed by explosion is incorrect, that it is contradicted by data and even by his own model. He could have then used his expertise to suggest new directions for the demographic and economic research.

The same applies to Galor. He uses Maddison's data but surprisingly he never attempts to analyse them. He prefers to distort them (Galor, 2005a, 2005b, 2007, 2008a, 2008b, 2008c, 2010, 2011, 2012a, 2012b, 2012c; Galor & Moav, 2002) to support the preconceived but erroneous ideas. He knows mathematics and he should be familiar with hyperbolic distributions. If he analysed data, the same data that he used in his publications, he would have soon discovered that the established knowledge in demography and in economic research is scientifically unsupported. He could have then also used his expertise to suggest new lines of research. These examples show how strongly the established knowledge is established and that even prominent researchers can be easily misled by the system of its doctrines.

5. Camouflaging the hyperbolic equation

Here is an example how the well-known differential equation describing hyperbolic growth was disguised as something new, which was supposed to explain the mechanism of growth based on the assumption that the growth of population is finely-tuned to the technological development. In its undisguised form, the differential equation (2) describes hyperbolic growth but does not explain its mechanism. It is just a mathematical equation, which when solved produces hyperbolic distribution. However, in its disguised form it seems to contain an explanation of the mechanism of growth. It seems to show that the growth of population is determined by the level of technology or knowledge.

This is a good example, which demonstrates that one should never be mesmerised by complicated mathematics. Mathematical formulations can be complicated and useful but just because they are complicated it does not mean that they are useful. Unified Growth Theory (Galor, 2005a, 2011), which is supposed to explain the mechanism of economic growth, is full of such complicated mathematical formulations. However, these complicated formulae do not explain anything. They just translate erroneous concepts into mathematical language. Data describing economic growth and the growth of population (Maddison, 2001) were used but they were never analysed to check the proposed theory. They were presented in a distorted way to make the impression that theory is confirmed by data. In the example presented here, the discussed mathematical equations are relatively simple and even attractive but they give a corrupted and mathematically unacceptable representation of the well-known differential equation [eqn (2)] describing hyperbolic growth.

Korotayev (2005) used the following differential equations to describe and explain the growth of population:

$$\frac{dS(t)}{dt} = a[bK(t) - S(t)]S(t), \tag{47}$$

$$\frac{dK(t)}{dt} = cS(t)K(t). \tag{48}$$

According to his interpretation "*K* is the level of technology/knowledge, *bK* corresponds to the number of people (*N*) [$S(t)$ in our notation], which the earth can support with the given level of technology (*K*)" (Korotayev, 2005, p. 81). Thus *bK* is interpreted as the carrying capacity of the planet.

To fit the population data, Korotayev carried out step-by-step calculations based on the eqns (47) and (48) but presented in a different form:

$$K_{i+1} = K_i + cS_iK_i, \tag{49}$$

$$S_{i+1} = S_i + a(bK_{i+1} - S_i)S_i. \tag{50}$$

There is absolutely no reason why $K(t)$ should represent technology or knowledge. We can call it whatever we want but just because we call it technology, knowledge or the carrying capacity it does not mean that it represents these imposed by us concepts. In the logistic model, which is similar to the eqn (47), it is a constant describing the limit to growth, which may or may not represent the carrying capacity. However, we shall show that in eqns (47) and (48), $K(t)$ has nothing to do even with the limit to growth. It is a variable that does not restrict growth in any way because $K(t)$ is in fact $S(t)$. It is simply the size of population or the size of any, hyperbolically-increasing quantity. Consequently, even if we use this set of differential equations and even if we fit data, we cannot claim that we have *explained* the growth of human population.

To show that $K(t)$ is in fact just $S(t)$, let us start with the differential equation for the hyperbolic growth [see eqn (2)]:

$$\frac{dS(t)}{dt} = kS^2(t). \tag{51}$$

It is the same equation as eqn (2) but it is now presented in s slightly different form. Let us now replace *k* by

$$k \equiv c \equiv a(b-1). \tag{52}$$

where *c*, *a* and *b* are constants. Mathematically, this modification is acceptable because *k* is a constant and we can always replace a constant by any combination of constants. Normally, we would not do it. We do it here to show that the eqns (47) and (48) represent a complicated representation of the eqn (51). However, we shall show that these equations represent also a corrupted form of the eqn (51).

Equation (51) can now be expressed as

$$\frac{dS(t)}{dt} = a[bS(t) - S(t)]S(t). \tag{53}$$

This equation is already almost the same as the eqn (47). But now let us corrupt this equation. Let us replace one $S(t)$ in the eqn (53) by $K(t)$, while keeping the other $S(t)$ unchanged. So now we have two equations:

$$\frac{dS(t)}{dt} = a[bK(t) - S(t)]S(t), \tag{54}$$

$$K(t) = S(t). \tag{55}$$

If $K(t) = S(t)$ then of course:

$$\frac{dK(t)}{dt} = \frac{dS(t)}{dt}. \tag{56}$$

However, according to the eqns (51), (52) and (55), and supported by the selective treatment of $S(t)$, we have

$$\frac{dS(t)}{dt} = kS^2(t) = cS^2(t) = cS(t)K(t). \tag{57}$$

So finally, we now have

$$\frac{dS(t)}{dt} = a[bK(t) - S(t)]S(t), \tag{58}$$

$$\frac{dK(t)}{dt} = cS(t)K(t). \tag{59}$$

These two equations are precisely the same as the eqns (47) and (48), and functionally the same as the eqn (51). However, now we have three constants, *a*, *b* and *c*, rather than just one constant *k*. We also have one $S(t)$ disguised as $K(t)$, while the other $S(t)$ retains its identity. *The variable* $K(t)$ *is just the size of the population*. It has nothing to do with technology, knowledge or carrying capacity. Korotayev's differential equations do not explain the mechanism of growth. They only describe the growth of human population using the well-known mathematical differential equation for the hyperbolic growth. They do not explain why the growth of population was hyperbolic.

We have repeated the calculations of Korotayev (2005) using his eqns (47) and (48) and his step-by-step procedure defined by eqns (49) and (50). Results are presented in Figure 4. They show that $K(t)$ is precisely the same as $S(t)$, $K(t) \equiv S(t)$. The two distributions are indistinguishable. $K(t)$ is not technology, knowledge or carrying capacity but the size of the hyperbolically increasing quantity, such as population or the GDP.

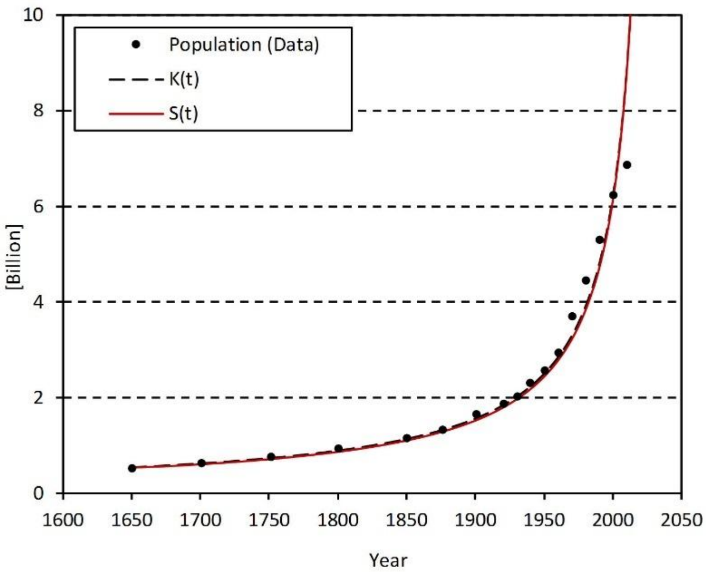

Figure 4. *Results of calculations carried out using eqns (47) and (48) and the step-by-step procedure defined by eqns (49) and (50). They confirm that* $K(t) \equiv S(t)$. *Eqns (47) and (48) represent a camouflaged eqn (51), which describes hyperbolic growth. The data represent the average values of the size of the world population calculated using the compilations of Manning (2008) and of the US Census Bureau (2016).*

Korotayev accepts now that he made a mistake: "I agree with what you wrote." (Korotayev, 2015). However, his model and his calculations have been published in a peer-reviewed journal and as far as we can tell they were never corrected.

This earlier attempt by Korotayev (2005) was followed by a new approach designed to link the growth of population with economic growth (Korotayev & Malkov, 2012; Korotayev, Malkov & Khaltourina, 2006a):

$$\frac{dS(t)}{dt} = aq(t)S(t), \tag{60}$$

$$\frac{dq(t)}{dt} = bq(t)S(t), \tag{61}$$

where $S(t)$ is again the size of human population, a and b are adjustable constant and $q(t)$ is claimed to be, again for no convincing reason, the surplus of the GDP per capita.

If we compare eqns (51) and (60), we can see that if the eqn (60) is supposed to describe hyperbolic growth of population or the GDP, then $q(t)$ cannot be anything else but $S(t)$, the size of the population or the GDP. The eqn (60) is the same as the eqn (51) except that, for no good reason, one $S(t)$ is now replaced by $q(t)$. However, this also means that $a = b$ and indeed the authors of these two equations have determined that $a = 1.04b$, which is as good as $a = b$. The two equations are *identical*. They are not two different equations but the same equation repeated twice, the same equations as eqn (51) but now one $S(t)$ is again disguised, this time as $q(t)$, which for absolutely no convincing reason is called the surplus of the GDP per capita.

We can replace *S* by any letter in the alphabet. We can call the replaced letter anything we want but in this context, it is nothing else but the size of population or the GDP or the size of any other hyperbolically increasing quantity. We are back to the original habit of corrupting the perfectly good and legitimate hyperbolic equation, but now we are not representing one of the $S(t)$ as $K(t)$, which for no good reason was called technology or knowledge. We are now representing one of the $S(t)$ as $q(t)$, which again for no convincing reason is called the surplus of the GDP per capita.

In the earlier mistake, the hyperbolic differential eqn (51) or (2) was disguised as two distinctly different equations. Now it is disguised as two similar equations, which are in fact identical. Previously, the growth of population was supposed to have been explained by technology, knowledge or the carrying capacity, which was incorrect and misleading, because the so-called technology or knowledge or the carrying capacity was nothing else but the size of the hyperbolically increasing quantity $S(t)$. Now, the growth of population is supposed to be explained by the surplus of the GDP per capita, which is again incorrect and misleading because the claimed surplus of the GDP per capita is just $S(t)$, which represents the size of the hyperbolically increasing quantity. They are making the same mistake as before. They have not introduced any new idea but present the same mistake in a different mathematical form.

The next step is to make it all even more mysteriously complicated. For obscure reasons, the growth of human population is now supposed to be described by a set of three differential equations (Khaltourina & Korotayev, 2007; Korotayev, Malkov & Khaltourina, 2006a, 2006b):

$$\frac{dS}{dt} = aqS\left(1 - L\right), \tag{63}$$

$$\frac{dq}{dt} = bqS, \tag{64}$$

$$\frac{dL}{dt} = cqL\left(1 - L\right), \tag{65}$$

where *a*, *b* and *c* are adjustable constants and *L* is claimed, without any convincing justification, to represent the fraction of *literate* population, which implies that $1 - L$ is the fraction of the illiterate population (Korotayev, 2015). In these equations, the time dependence is not explicitly displayed. So, $S \equiv S(t)$, $q \equiv q(t)$ and $L \equiv L(t)$

Again, if the eqn (63) is supposed to describe hyperbolically increasing distribution, such as population or the GDP, then $q(1 - L) \equiv S$. We can replace one *S* in the hyperbolic differential equation (51) by whatever we want but functionally it will be still *S*.

A modified version of the three equations (63)-(65) are equations containing even more, spurious and meaningless parameters (Korotayev, Malkov & Khaltourina, 2006b):

$$\frac{dS}{dt} = aq^{\varphi_1} S^{\varphi_2} \left(1-L\right)^{\varphi_3}, \tag{66}$$

$$\frac{dq}{dt} = bq^{\varphi_4} S^{\varphi_5}, \tag{67}$$

$$\frac{dL}{dt} = cq^{\varphi_6} L^{\varphi_7} \left(1-L\right)^{\varphi_8}, \tag{68}$$

where φ_i with $i = 1-8$ are arbitrary adjustable positive constants "not necessarily equal to one" (Korotayev, Malkov & Khaltourina, 2006b, p. 73). The interpretation of these additional parameters is also obscure.

Korotayev and his associates claim that they can generate hyperbolic growth with a transition to a new type of growth. However, they did not introduce any new concepts, which could justify this claim. They have just replaced two equations by three and one spurious variable by two. They follow the same idea as expressed in the eqns (47) and (48). In the original equations, a spurious variable $K(t)$, was introduced which for no good reason was called technology or knowledge or the limit to growth and which turned out to be just the size of population or some other hyperbolically increasing quantity. Now, the original two equations are replaced by three because two spurious variables are introduced, $q(t)$ and $L(t)$, which for no convincing reason are called the surplus of the GDP per capita and the fraction of literate population, respectively. The method of calculations is also the same, i.e. as outlined in the eqns (49) and (50).

Whatever is done is hidden in the obscure calculation procedure. As before, one would have to repeat their calculations to understand better the source of error or maybe to become convinced that whatever they are doing is correct. However, from a start, there is no convincing justification for claiming that q represents the surplus of the GDP per capita and that L represents the fraction of literate population, described also as "potential teachers" (Korotayev, Malkov & Khaltourina, 2006a, p. 26, 2006b, p. 73). There is also no convincing justification for claiming that the growth of population should be so vitally dependent on the surplus of the GDP per capita and on the number of potential teachers.

We could probably invent many other complicated formulae to replace the simple and working eqn (2) or (51). We could also label the new introduced variables or constants in whatever way we want but could we claim that we have contributed to a better understanding of the mechanism of hyperbolic growth?

6. Microscopic growth theory

The concept of Karev (2005a, 2005b, 2010) and Karev & Kareva (2014) is to see human population (or other biological systems) as being made of individuals, each characterised by a certain, unique parameter a. In a more general formulation of this theory, this uniquely defining parameter is a multi-dimensional vector $\vec{a} = (a_1, a_2 ... a_n)$ made of many characteristic components. In the extreme case, we could think that the components of the vector $\vec{a}$ are made of genes or even of the components of the whole genome. In such a case, the multidimensional vector would be made of 10^6-10^9 components (Karev, 2005b).

This theory is based on the advanced and aesthetically appealing mathematics. We shall explain the fundamental concepts of this theory. Once the fundamental ideas are understood, it will be easier for anyone to read the more advanced description presented by Karev (2005a, 2005b, 2010) and Karev & Kareva (2014). In our discussion, we shall replace vector $\vec{a}$ by constant *a*.

Rather than dealing with individuals characterised by the parameter *a*, it is assumed that the entire population is made of a certain number of groups of *a*-clones, each group characterised by the same parameter *a* and each group made of $n(t,a)$ number of members at a given time *t*. In order to calculate the growth of population we first calculate the growth of each group of *a*-clones. The differential equation describing the growth of *a*-clones is given by

$$\frac{1}{n(t,a)}\frac{dn(t,a)}{dt} = F(t,a). \tag{69}$$

The function $F(t,a)$ is called "the per capita reproductive rate" (Karev & Kareva, 2014, p. 73) but the well-known and accepted definition of the net reproductive rate is the number of daughters born per woman in her lifetime. In the same publication, $F(t,a)$ appears also as $ag(N)$, where in our notation N represents $S(t)$, and $g(N)$ is interpreted as "some function, chosen depending on the specifics of the model" (Karev & Kareva, 2014., p. 69). Karev agrees (Karev, 2015) that it would be better to call $F(t,a)$ simply as a *growth factor*, which will depend on the model used in the calculations. However, if we use the concept of the general law of growth (Nielsen, 2016k), then this factor can be identified simply as the force of growth, which in the microscopic theory can have a variety of representations.

The factor $F(t,a)$ contains all the information about the mechanism of growth of each group of *a*-clones. The microscopic theory does not describe any single mechanism but gives a complete freedom to explore a variety of options. Each specifically chosen mathematical representation of the factor $F(t,a)$ will describe a certain mechanism of growth of each group of *a*-clones, but the mechanism will remain unknown until the chosen mathematical description of $F(t,a)$ is not only convincingly explained but also justified.

The additional complication in this theory is that the calculated size $S(t)$ of the population made of numerous groups of clones will depend on how their growth is *combined*. To understand the mechanism of growth of population it is necessary to explain not only the factor $F(t,a)$ but also to justify a specific mathematical way of combining the growth of all clones.

The growth rate of population is given by:

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = E(t)F(t,a), \tag{70}$$

where $E(t)$ is a function describing the mathematical way of combining the growth of population in all groups of clones. So now, the description of the mechanism of growth depends not only on $F(t,a)$ but also on $E(t)$. In order to explain this mechanism, it is not enough to explain and justify the factor $F(t,a)$ but also $E(t)$. The force of growth is given by the product of $F(t,a)$ and $E(t)$.

The calculation of $E(t)$ is based on the assumption that the populations of various groups of clones are distributed along a certain probability density function $p(t,a)$ defined as:

$$p(t,a)=\frac{n(t,a)}{S(t)}. \tag{71}$$

The function $p(t,a)$ describes the probability of having $n(t,a)$ number of individuals characterised by the unique parameter *a*, i.e. the probability distribution of the parameter *a*.

The definition of $E(t)$, based on the publication of Hofbauer and Sigmund (1998), is:

$$E(t)=\int_0^\infty ap(t,a)da. \tag{72}$$

To illustrate the application of this theory to the description of the growth of human population we shall use three models of growth presented by Karev (2005a) leading to three solutions.

6.1. Solution 1

This solution is based on the assumption that $F(t,a)=a$. Consequently,

$$\frac{1}{n(t,a)}\frac{dn(t,a)}{dt}=a. \tag{73}$$

In this model, it is assumed that each group of *a*-clones increases exponentially. The growth is prompted by a constant force generating a constant growth rate. This is the force of unknown nature. We do not know why this force should be constant. We just assume that it is. Thus, from the very beginning we cannot explain the mechanism of growth. Whatever we shall calculate will not help us to understand the growth of population. Maybe we shall be able to fit the data but we already know that the data can be fitted well (Nielsen, 2016a, 2016c) using the simple expression describing hyperbolic distribution [see eqn (1)]. The approach proposed by the microscopic theory will offer an alternative description but it is more complicated and there is no clear reason for preferring this approach.

The growth of human population as a whole is given now by:

$$\frac{1}{S(t)}\frac{dS(t)}{dt}=E(t)a. \tag{74}$$

Karev (2005a) gives the following expression for $E(t)a$, determined by his choice to describe mathematically the probability density function $p(t,a)$:

$$E(t)a=\eta+\frac{k}{s-t}, \tag{75}$$

where η, *k* and *s* are adjustable constants (*s*, *k*>0, $-\infty < \eta < \infty$). For $t=s$, this function escapes to infinity.

The differential equation for the growth of human population is now given by:

$$\frac{1}{S(t)}\frac{dS(t)}{dt}=\eta+\frac{k}{s-t}\,. \tag{76}$$

The right-hand side of this equation is again the force of growth of unknown origin and it is even less acceptable than the constant force because it is more complicated. If we had reservation about using a constant force of unknown origin to describe the growth of a group of clones [eqn (73)] our reservation is now even stronger because the force describing the growth of population is significantly more complicated and also unexplained. We can see that explaining the mechanism of growth is becoming progressively more difficult. We might only hope that perhaps our formula will give a better description of data but we shall soon see that it does not.

The solution of the eqn (76) presented by Karev (2005a) is

$$S(t)=S_0\frac{e^{\eta t}}{(1-t/s)^k}\,,\ \text{for } t<s\,, \tag{77}$$

which is the exponentially modulated hyperbolic-like growth because it increases to infinity when $t=s$. It is not clear why we should want to use this distribution when we already have a simpler distribution given by the eqn (1) fitting the population data.

If $\eta = 0$, then

$$S(t) = S_0\frac{1}{(1 - t / s)^k}\ \text{for } t < s\,. \tag{78}$$

The size of population approaches singularity when time t approaches the parameter s. For $k=1$it is the first order hyperbolic growth given by the eqn (1). We can explain this formula but we cannot explain the mechanism of growth. We cannot explain why the growth should be expected to behave in this particular way.

6.2. Solution 2

Solution 2 is also based on the assumption of an exponential growth of each group of *a*-clones but now a different mathematical description is used for the probability density function $p(t,a)$, which gives different expression for $E(t)a$ used in the eqn (74):

$$E(t)a=\frac{1}{s-t}+\frac{c}{1-e^{c(s-t)}}\,. \tag{79}$$

$E(t)a$ escapes to infinity when $t=s$.

The differential equation for the growth of human population is now given by:

$$\frac{1}{S(t)}\frac{dS(t)}{dt}=\frac{1}{s-t}+\frac{c}{1-e^{c(s-t)}}\,, \tag{80}$$

and its solution by:

$$S(t) = S_0 \frac{1 - e^{c(t-s)}}{(1 - t/s)(1 - e^{-sc})}. \quad (81)$$

Parameters used by Karev (2005a) are $c \approx 0.114$ and $s = 2026$. The corresponding product cs is large and the second term in the denominator can be neglected. The formula (81) can now be presented as

$$S(t) = S_0 \frac{1 - e^{c(t-s)}}{(1 - t/s)}. \quad (82)$$

This solution resembles the first-order hyperbolic growth because the denominator is a linear function of *t*, and if not for the function appearing in the numerator, the growth of the population would escape to infinity at $t = s$. However, when $t = s$, the numerator is also zero. Close examination of the eqn (82) shows that when *t* approaches *s*, this fraction approaches a constant value, which depends on parameters *s* and *c*. Furthermore, calculations show that for $t < s$, $S(t)$ increases approximately hyperbolically but for $t > s$, it increases approximately exponentially. Thus, the Solution 2 can be seen as being made of two parts: a hyperbolic growth to $t \approx s$ and an exponential growth from $t \approx s$ with an instantaneous discontinuity at precisely $t = s$.

Mathematically, this formula is interesting because it shows that by assuming a certain force of growth it might be possible to generate a trajectory, which would, at a certain stage, change from hyperbolic to a different type of growth. If we could explain the nature of this peculiar force and if we could reproduce data, we would make a huge progress in the understanding of the mechanism of growth. However, in this particular case we have no clue about the nature of this peculiar force and, as we shall soon see, the formula given by the eqn (82) does not fit the data.

6.3. Solution 3

Solution 3 is based on the assumption that $F(t,a)$, which in Solutions 1 and 2 was constant, is now represented by the modified logistic growth rate (Gilpin & Ayala, 1973).

$$F(t,a) = a\left[1 - \left(\frac{S(t)}{K}\right)^k\right]. \quad (83)$$

Again, we do not know the nature of this force.

Under this assumption, the growth of each group of *a*-clones is given by

$$\frac{1}{n(t,a)}\frac{dn(t,a)}{dt} = a\left[1 - \left(\frac{S(t)}{K}\right)^k\right], \quad (84)$$

where $k = const > 0$ and K is the limit to growth.

Unless $k = 1$, the driving force of growth for each group of clones decreases non-linearly with the size $S(t)$ of the whole population. The growth of each group of clones is no longer defined by the parameter *a* alone, which represents exclusive characteristics of any particular group of clones, but it also depends on the size of the *whole* population. The growth of each group of clones is somehow coupled to the growth of other clones.

The differential equation for the whole population is now given by:

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = E(t)a\left[1-\left(\frac{S(t)}{K}\right)^k\right]. \tag{85}$$

Karev (2005a) uses the following expression for $E(t)a$:

$$E(t)a = \frac{1}{s-p(t)} + \frac{1}{1-e^{c[s-p(t)]}}, \tag{86}$$

where $p(t)$ is a solution of Cauchy problem:

$$\frac{dp(t)}{dt} = 1 - \left\{S_0 \frac{s}{s-p(t)} \frac{1-\exp\{c[p(t)-s]\}}{1-Ke^{-sc}}\right\}^k. \tag{87}$$

So, now, the differential equation describing the growth of human population is given by:

$$\frac{1}{S(t)}\frac{dS(t)}{dt} = \left[\frac{1}{s-p(t)} + \frac{1}{1-e^{c[s-p(t)]}}\right]\left[1-\left(\frac{S(t)}{K}\right)^k\right], \tag{88}$$

and the size of population by

$$S(t) = S_0 \frac{s}{s-p(t)} \frac{1-\exp\{c[p(t)-s]\}}{1-e^{-sc}}. \tag{89}$$

The description of the growth of human population is now significantly more complicated. Solutions given by eqns (78) and (82) were relatively simple because they were based on the assumption of the simplest type of growth of the individual groups of clones, growth of each clone prompted by a constant force. Even though we were not able to explain the mechanism of growth of the entire population made of various groups of clones we could at least explain the mathematical formulae describing growth. However, in the case of the growth described by the eqn (89) we cannot even understand this formula let alone to understand the mechanism of growth of the entire population. We do not understand why the growth of human population should follow this particular trajectory. Even if we could fit the theory to data precisely and over the entire range of time we would be still unable to explain the mechanism of growth.

6.4. Comparing theory with data

Solutions 1-3 are shown in Figures 5 - 7. They are compared with data coming from a wide range of sources compiled by Manning (2008) and by the US Census Bureau (2016).

In Figure 5 we show the reciprocal values of data and the reciprocal values of Solutions 1-3. The advantage of using this display is that the decreasing linear trends identify uniquely hyperbolic distributions (Nielsen, 2014).

The common feature of all these solutions is that over the nearly entire range of time during the AD era they all follow hyperbolic trajectories. However, they reproduce data over a strongly limited range of time. Consequently, there is no advantage in using these solutions. The microscopic theory does not give a better description of data than the simple hyperbolic formula, which can reproduce data over the past 12,000 years (Nielsen, 2016a). Solutions 2 and 3 are indistinguishable in this display. Solution 1 is only slightly different. Differences between these three solutions can be observed only towards the end of the time scale, as shown in Figures 6 and 7.

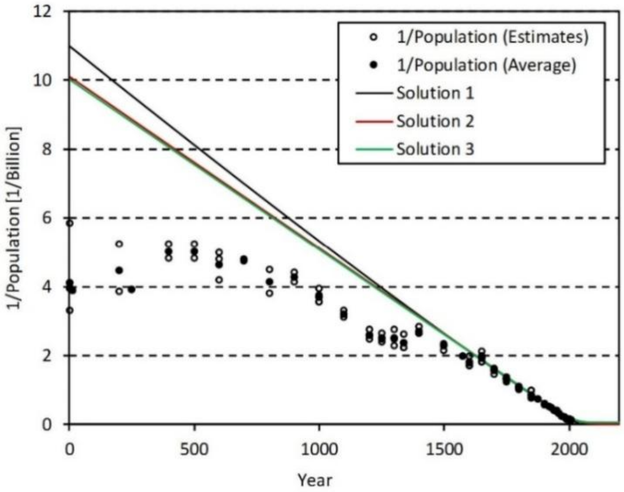

Figure 5. The decreasing straight lines of reciprocal values identify uniquely hyperbolic growth. *Reciprocal values of solutions 1, 2 and 3 [eqns (78), (82) and (89)] are compared with the reciprocal values of the world population data as compiled by Manning (2008) and by the US Census Bureau (2016).*

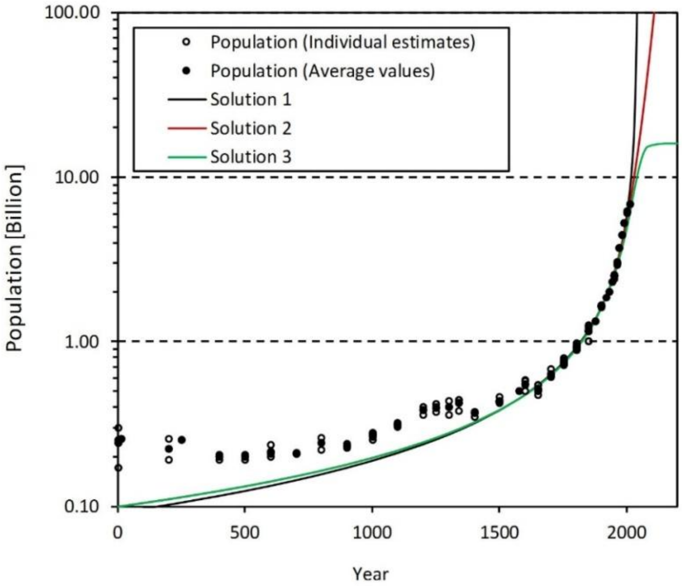

Figure 6. *Solutions 1, 2 and 3 [eqns (78), (82) and (89)] are compared with the world population data compiled by Manning (2008) and the by US Census Bureau (2016).*

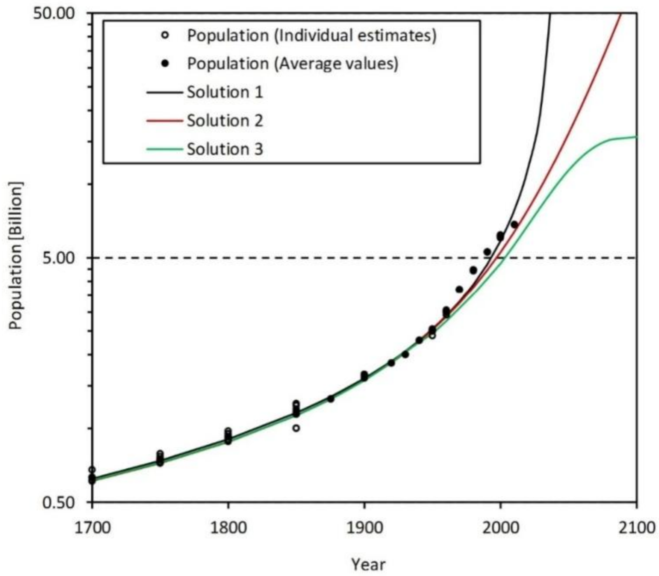

Figure 7. *Trajectories generated by Karev's Solutions 1, 2 and 3 (Karev, 2005a) in the region where they start to divert to different trajectories are shown together with population data using the compilations of Manning (2008) and of the US Census Bureau (2016). Solution 1 escapes to infinity. Solution 2 converts to an exponential growth, while Solution 3 converts into a logistic growth.*

While the concept of the microscopic approach to the study of the growth of population is interesting, it is not only extremely complicated but also it creates serious problems for the explanation of the mechanism of growth. Examples used by Karev (2005a) show that the complicated mathematical solutions generated by this theory imitate hyperbolic distributions, which can be represented by a much simpler equation [see eqn (1)]. Furthermore, these solutions reproduce only a very small range of data.

The problem with using this theory to explain the mechanism of growth is well illustrated by Solutions 1, 2 and 3 given by the eqns (78), (82) and (89) and by the accompanying expressions for $aE(t)$ given in the eqns (75), (79) and (86). While we can explain some of these expressions, we cannot use them to explain the mechanism of growth.

An interesting feature of this exercise is that a single force of growth can describe a trajectory, which at a certain stage can change from hyperbolic to some other type. If we could find a force that could reproduce data over the whole range of time and if we could explain the nature of this force, we would have made a huge progress in explaining the mechanism of growth. However, examples presented by Karev indicate that finding such a force of growth and explaining its origin is close to impossible.

7. Summary and conclusions

We have presented here a brief survey of attempts to understand hyperbolic distributions. The common characteristic of all these attempts is that they are not only complicated but that they are also unnecessarily complicated because a simple expression given by eqn (1) describes data exceptionally well (Nielsen, 2016a, 2016c). This simple formula describes not only the growth of population but also economic growth as expressed by the Gross Domestic Product (Nielsen, 2016b). Furthermore, by using this simple formula we can also easily describe income per capita and explain its puzzling features (Nielsen, 2015, 2016g).

Complicated methods used in the interpretations of hyperbolic growth did not yet result in explaining its mechanism. They also did not produce a better description of data than the descriptions given by the simple expression represented by the eqn (1).

When mathematical formulations become increasingly complicated it is usually a warning sign that we are on the wrong track, that we should stop, regroup and look for simpler descriptions and solutions. A simple formula [eqn (1)] describing population and economic growth suggests that there must be also a simple explanation of their mechanisms. Such a simpler explanation will be proposed in the next publication.

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