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Book

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REAL-TIME FORECAST OF DSGE MODELS WITH TIME-VARYING VOLATILITY IN GARCH FORM

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Abstract

Recent research shows that time-varying volatility plays a crucial role in nonlinear modeling. Contributing to this literature, we suggest a DSGE-GARCH approach that allows for straight-forward computation of DSGE models with time-varying volatility. As an application of our approach, we examine the forecasting performance of the DSGE-GARCH model using Eurozone real-time data. Our findings suggest that the DSGE-GARCH approach is superior in out-of-sample forecasting performance in comparison to various other benchmarks for the forecast of inflation rates, output growth and interest rates, especially in the short term. Comparing our approach to the widely used stochastic volatility specification using in-sample forecasts, we also show that the DSGE-GARCH is superior in in-sample forecast quality and computational efficiency. In addition to these results, our approach reveals interesting properties and dynamics of time-varying correlations (conditional correlations).

JEL Classifications: C32; E30; E37.

Keywords: DSGE; forecasting; GARCH; stochastic volatility; conditional correlations.

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models have become standard toolkits of macroeconomic analysis used by central banks and other institutions. Because they allow for microeconomic foundations in their building blocks, these models allow researchers to overcome the Lucas critique (Lucas (1976)) and provide a forecasting quality close to or better than autoregressive models (see among others Adolfson *et al.* (2011), Rubaszek & Skrzypczyński (2008) and Del Negro & Schorfheide (2013)).

While these results are based on linearized DSGE models, recent studies such as Pichler (2008) or Ivashchenko *et al.* (2020) show that adding non-linearities to the modeling framework can further improve forecasting performance. This potential advantage of adding non-linearities however is usually overshadowed by increasing computational costs.

One way in which previous studies introduced non-linearities into the DSGE framework is by means of time-varying volatilities (see e.g. Sims & Zha (2006), Justiniano & Primiceri (2008), Bloom (2009), and Fernández-Villaverde & Rubio-Ramírez (2010)). As shown by Ivashchenko *et al.* (2020), considering this approach is relevant because, even with a first-order approximation, models in which time-variation of volatilities is introduced are able to produce superior forecasts in comparison to their non-varying counterparts[1].

Previously, authors such as Diebold *et al.* (2017) implemented procedures for estimating DSGE models with time-varying volatility and for evaluating their forecasting performance in comparison to benchmark constant volatility DSGE models. In this work we introduce an alternative approach for the modeling of time-varying volatility which is based on expressing volatility as a GARCH process. To the best of our knowledge, the only other study to consider a DSGE model with GARCH volatilities is Andreasen (2012) who studies the effects of rare disasters and uncertainty shocks on risk premia in a New Keynesian DSGE model approximated to second and third order, using usual perturbation methods. The approach considered in Andreasen (2012) and our approach are complementary as we use real-time data and test for the forecasting performance of the model in comparison to their constant variance counterparts and other time-varying specifications. However, the computational approaches are quite different.

Using real-time data for the euro area, we first build a model of the economy and use various specifications for the modeling of the volatility process. Among the specifications we compare, DSGE with GARCH(1,1), DSGE with Random-Walk GARCH and DSGE with constant volatility, we show that the GARCH(1,1) specification is mostly superior in producing out-of-sample point and density forecasts. We then compare the GARCH(1,1) specification to the widely used stochastic volatility model as utilized in Justiniano & Primiceri (2008) or Diebold *et al.* (2017) within DSGE frameworks. Using in-sample forecasts and examining inefficiency factors, we demonstrate that the GARCH specification produces better forecasts and is computationally more efficient in comparison to the stochastic volatility approach which is based on Gibbs sampling and Gaussian-mixture approximation of density.

Our contribution supports the view that incorporating time-varying volatilities is important in macroeconomic modeling, as highlighted in previous research such as Fernández-Villaverde & Rubio-Ramírez (2010). Using various metrics, we further show that the GARCH specification is superior to other time-varying volatility specifications and exhibits computational costs similar to those of linearized DSGE models.

We proceed as follows. In section 2 we describe the DSGE model, data and priors, our DSGE-GARCH approach and corresponding estimation procedure, in section 3, we present our results and compare the DSGE-GARCH approach to the widely used stochastic volatility (DSGE-SV) approach. In the last section, we present our conclusions.

[1]It is important to mention that despite providing a superior forecast, the first-order approximation of a DSGE model solution is not affected by the variance matrix (see e.g. Justiniano & Primiceri (2008) or Diebold *et al.* (2017)) and hence allows for the computation of a special approximation that reflects time-variation effects with computational costs similar to a usual linear first-order approximation.

2 Model and Estimation

2.1 Model, data and measures

As a benchmark, we use the small-scale model presented in Diebold *et al.* (2017), which is a variant of the Smets *&* Wouters (2007) model without capital accumulation, wage stickiness and habit formation. In the following, we present key features of the model. The consumption Euler equation can be represented by

$$c_t = E_t(c_{t+1} + z_{Y,t+1} - (r_t - p_{t+1})/\tau) \tag{1}$$

where c_t is consumption, r_t is the nominal interest rate, p_t is the inflation rate, τ is the relative degree of risk aversion and $z_{Y,t}$ is a term that describes the evolution of the technology shock. The New-Keynesian Phillips curve is given by

$$p_t = \frac{p_{t-1}\iota}{1+\iota\beta} + \frac{\beta E_t p_{t+1}}{1+\iota\beta} + \frac{(c_t + y_t v_1)(1-\chi\beta)(1-\chi)}{(\chi^*(1+\iota\beta))} \tag{2}$$

where ι is the share of price setters that index their prices to lagged inflation, χ is the probability that governs the ability of price setters to reset their prices, v_1 is the Frisch elasticity of labor supply and β is the representative household's discount factor. Output consists of household consumption c_t and government consumption g_t:

$$y_t = c_t + g_t \tag{3}$$

The central bank follows a monetary policy rule that includes interest rate smoothing, and sets nominal interest rates r_t in response to deviations of inflation and output growth from their respective targets:

$$r_t = \gamma_r r_{t-1} + (1-\gamma_r)(\gamma_{rP}(p_t - \overline{p}) + (1-\gamma_{rP})(p_{*,t} - \overline{p}_*) + \gamma_{rY}(y_t - y_{t-1} + z_{Y,t})) + z_{R,t} \tag{4}$$

Here, γ_r is the interest smoothing parameter and $z_{R,t}$ is the monetary policy shock. The inflation target p_t^* and government consumption g_t evolve according to the following exogenous processes:

$$p_{*,t} = \gamma_{PT} p_{*,t-1} + z_{PT,1} \tag{5}$$

$$g_t = \gamma_{GT} g_{t-1} + z_{G,1} \tag{6}$$

Here, $z_{G,t}$, $z_{PT,t}$ and $z_{R,t}$ are zero mean iid exogenous processes, while $z_{Y,t}$ is an AR(2) process with the following parameterization:

$$z_{Y,t} = \eta_{1,Y}(1-\eta_{2,Y})z_{Y,t-1} + \eta_{2,Y} z_{Y,t-2} + (1 - \eta_{1,Y}(1-\eta_{2,Y}) - \eta_{2,Y})\eta_{0,Y}\varepsilon_{Y,t} \tag{7}$$

The observed variables that link measurement equations to observables include the inflation rate $obs_{P,t}$, 10-year inflation expectations $obs_{P40,t}$, nominal interest rates $obs_{R,t}$ and the quarterly growth rate of GDP $obs_{Y,t}$.

$$obs_{P,t} = 400 \cdot (p_t + \pi) \tag{8}$$

$$obs_{P40,t} = 400 \cdot E_t\left(\pi + \sum_{i=1}^{40} p_{t+i}/40\right) \tag{9}$$

$$obs_{R,t} = 400 \cdot (\pi + \gamma - \log(\beta) + r_t) \tag{10}$$

$$obs_{Y,t} = 100 \cdot (\gamma + y_t - y_{t-1} + z_{Y,t}) \tag{11}$$

Priors

For priors, we use those that were used by Diebold *et al.* (2017), which are based on Smets *&* Wouters (2007).

Table 1: Prior distributions

	Density	Mean	std
stderr ε_G	I-G	0.001	0.02
stderr ε_{PT}	I-G	0.001	0.02
stderr ε_R	I-G	0.001	0.02
stderr ε_Y	I-G	0.001	0.02
τ	N	1.5	0.37
v_l	G	2	0.75
ι	B	0.5	0.15
χ	B	0.5	0.1
γ_{rP}	N	1.5	0.25
γ_{rY}	N	0.12	0.05
400log$(1/\beta)$	G	1	0.4
400π	G	2.48	0.4
100γ	N	0.4	0.1
γ_r	B	0.5	0.2
γ_{GT}	B	0.5	0.2
γ_{GT}	B	0.5	0.2
$\eta_{2,Y}$	U	0	0.67
γ_{PT}	B	0.5	0.2
$\eta_{R,G}$	B	0.6	0.25
$\eta_{R,PT}$	B	0.6	0.25
$\eta_{R,Y}$	B	0.6	0.25
$\eta_{S,G}$	B	0.9	0.085
$\eta_{S,PT}$	B	0.9	0.085
$\eta_{S,Y}$	B	0.9	0.085

2.2 DSGE-GARCH

The first-order approximation of a model with a rational expectations solution can be formulated to have the following form:

$$\begin{aligned} Y_t &= HX_t + u_t \\ X_t &= A_X X_{t-1} + A_\varepsilon \varepsilon_t \end{aligned} \tag{12}$$

where Y_t represents the vector of observables, X_t represents the vector of endogenous variables and ε_t represents the vector of structural innovations. While this form can be solved with various algorithms such as those introduced in Blanchard *&* Kahn (1980), Collard *&*

Juillard (2001), Sims (2002) and or Schmitt-Grohé & Uribe (2004) to name a few, the solution remains the same and does not depend on the variance of shocks var(ε_t). Following this, we can introduce a process for the variance term that exhibits GARCH properties and does not influence the rational expectations solution approximation:

$$E(\varepsilon_t\varepsilon_t'|I_{t-1}) = E_{t-1}(\varepsilon_t\varepsilon_t') = V_t = V_c + \sum_{i=1}^{N\ GARCH} G_i V_{t-i} G_i' + \sum_{i=1}^{N\ ARCH} A_i E_{t-i}(\varepsilon_{t-i}\varepsilon_{t-i}')A_i' \quad (13)$$

Here, I_t is information available up to time t and V_c is the "constant" positive definite component of the variance V_t, thus making V_t positive definite too. A useful property of this formulation is that the use of a Kalman filter produces the values of $E_t(\varepsilon_t\varepsilon_t')$. Consequently, the likelihood can be computed directly and the Markov Chain Monte Carlo (MCMC) sampling procedure becomes computationally similar to a usual linearized DSGE model.

Similar to other models, it is straightforward to extend GARCH models to the multivariate case. An important difference of our definition of GARCH lies in the usage of conditional expectations of unobserved shocks. For state-space models with exact identification of shocks, similar to VAR models, conditional and unconditional expectations of shocks are equivalent whereas for asymptotically exact identification of shocks, (e.g. linearized DSGE models with an equal number of shocks and observables or other linear state-space models) conditional and unconditional shocks are asymptotically equivalent.

In the Kalman filter initialization of the state-space model presented in eq. 11, the variance of shocks V_0 and the variance of unobserved variables X_0 are used to compute the density of observed variables Y_t, which is normally distributed due to the linear nature of the equation describing observed variables. Then, the conditional density of unobserved variables $(X_t|I_t)$ and the conditional density of current shocks $(\varepsilon_t|I_t)$ are computed. The next step of the Kalman filter involves the computation of the conditional variance of shocks according to eq. 12 and differs from a conventional computation of a linear setup. Finally, the density of $(X_{t+1}|I_t)$ is computed, which is similar to the forecasting step of a Kalman filter. The parameterization we use suggests that the matrices G and A are diagonal and thus prevents an increase in the number of parameters.

For those cases when the information set I_t is sufficient for the identification of the shocks' values, we receive the standard multidimensional GARCH such that

$$E_t(\varepsilon_t\varepsilon_t') = \varepsilon_t\varepsilon_t'$$

However, in the case of DSGE models we often don't have sufficient observed variables and there is uncertainty about initial conditions. As a consequence, the usual Kalman Filter procedure produces conditional densities of shocks that are normally distributed such that

$$\varepsilon_t \sim N\Big(E_t(\varepsilon_t), \mathrm{var}_t(\varepsilon_t)\Big)$$
$$E_t(\varepsilon_t\varepsilon_t') = E_t(\varepsilon_t')(\varepsilon_t) + \mathrm{var}_t(\varepsilon_t)$$

While the conditional variance of shocks is usually small (when the number of shocks is equal to number of observed variables), it is not equal to zero.

DSGE specifications usually assume that innovations are uncorrelated while recent studies show that this assumption doesn't hold in most cases[2]. E.g. Cúrdia & Reis (2010) model U.S. business cycles by allowing structural shocks to be correlated, finding as a result that government spending and technology shocks are more relevant while changes in markups are less relevant than is typically found. A feature of our GARCH specification is that conditional correlations of shocks as implied by V_t can be non-zero even if V_C, A and G are

[2]See e.g. Falter *et al.* (2018) or Georgiadis & Jančoková (2020) for an elaboration on this point.

diagonal matrices. This is due to the multidimensional ARCH component of the GARCH process.

The filtering procedure is straightforward and the initial condition is usually given by the unconditional variance. However, the computation of this variance is not straightforward. To simplify this problem, we use a different parameterization, where the parameters define the unconditional variance of shocks, and V_c is computed according to the unconditional variance.

Lastly, there is an additional view on GARCH processes. GARCH is a special case of the stochastic volatility model. It is well known that ε_t^2 and ε_t are uncorrelated (in the case of normal distribution). This implies that $(\varepsilon_t)^2$, which affects volatility at time t is an uncorrelated volatility shock. Thus, eq. 13 can be rewritten as in eq. 14, with the additional term ξ denoting the "volatility" shock in eq. 15.

$$E_{t-1}(\varepsilon_t\varepsilon_t') = V_t = V_C + \sum_{i=1}^{N\ GARCH} G_i V_{t-i} G_i' + \sum_{i=1}^{N\ ARCH} A_i E_{t-i}(\varepsilon_{t-i}\varepsilon_{t-i}\ )A_i' = \quad (14)$$
$$= V_C + \sum_{i=1}^{N\ GARCH} G_i V_{t-i} G_i' + \sum_{i=1}^{N\ ARCH} A_i E_{t-i-1}(E_{t-i}(\varepsilon_{t-i}\varepsilon_{t-i}'))A_i' + \sum_{i=1}^{N\ ARCH} A_i \xi_{t-i} A_i' =$$
$$= V_C + \sum_{i=1}^{N\ GARCH} G_i V_{t-i} G_i' + \sum_{i=1}^{N\ ARCH} A_i V_{t-i} A_i' + \sum_{i=1}^{N\ ARCH} A_i \xi_{t-i} A_i'$$

$$\xi_t = E_t(\varepsilon_t\varepsilon_t') - E_{t-1}(\varepsilon_t\varepsilon_t') \tag{15}$$

Model Comparison

To see which volatility specification is superior, we compare three models: a constant volatility specification without GARCH parameters, a GARCH(1,1) specification and a Random-Walk GARCH specification (RW-GARCH). For the initialization of the variance of the GARCH and RW-GARCH specifications and their parameterization, we use a GARCH(1,1) which makes the specification in eq. 16 simpler. Also, instead of using the ARCH parameter η_{A^*} and the GARCH parameter η_{G^*}, we use their sum η_{S^*}, and their ratio η_{R^*} as described in eq. 17 and these can be transformed back easily according to eq. 18.

Estimating a constant variance version of the model given in eq. (1)-(11) results in monetary policy shocks that do not exhibit GARCH properties. Consequently, the GARCH(1,1) and RW-GARCH specifications are also modeled such that the monetary policy shock process has a constant variance.

$$E_{t-1}(\varepsilon_t\varepsilon_t') = V_t = V_C + G_1 V_{t-1} G_1' + A_1 E_{t-1}(\varepsilon_{t-1}\varepsilon_{t-1}')A_1' =$$
$$V_C + \begin{bmatrix} \eta_{G,G} & 0 & 0 & 0 \\ 0 & \eta_{G,PT} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{G,Y} \end{bmatrix} V_{t-1} \begin{bmatrix} \eta_{G,G} & 0 & 0 & 0 \\ 0 & \eta_{G,PT} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{G,Y} \end{bmatrix} \tag{16}$$
$$+ \begin{bmatrix} \eta_{A,G} & 0 & 0 & 0 \\ 0 & \eta_{A,PT} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{A,Y} \end{bmatrix} E_{t-1}(\varepsilon_{t-1}\varepsilon_{t-1}') \begin{bmatrix} \eta_{A,G} & 0 & 0 & 0 \\ 0 & \eta_{A,PT} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{A,Y} \end{bmatrix}$$

$$(\eta_{G^*})^2 + (\eta_{A^*})^2 = (\eta_{S^*})^2 \qquad (17)$$
$$\eta_{R^*} = \eta_{A^*}/\eta_{S^*}$$

$$\eta_{G^*} = \eta_{S^*}\sqrt{(1-\eta_{R^*}^2)} \qquad (18)$$
$$\eta_{A^*} = \eta_{S^*}\eta_{R^*}$$

For the RW-GARCH specification, the parameter η_{S^*} is equal to 1. The squared standard error parameters are used as the initial (unconditional) variance of shocks and are obtained by solving the expression in eq. 16 for the V_C matrix. The solution is given in eq. 19. In the case of the RW-GARCH model, the V_C matrix becomes a null matrix which implies a random walk without drift for the variance term.

$$\begin{bmatrix} (\text{std } \varepsilon_{G,t})^2 & 0 & 0 & 0 \\ 0 & (\text{std } \varepsilon_{PT,t})^2 & 0 & 0 \\ 0 & 0 & (\text{std } \varepsilon_{R,t})^2 & 0 \\ 0 & 0 & 0 & (\text{std } \varepsilon_{Y,t})^2 \end{bmatrix} = V = V_C + G_1 V G_1' + A_1 V A_1' \qquad (19)$$

It should be noted that while our priors are similar to the benchmark constant volatility model of Diebold *et al.* (2017), they deviate from the priors they used for their time-varying models because the unconditional variance of the SV-AR model utilized by these authors can be infinite. Correspondingly, we use priors such that the prior for the unconditional standard deviation of shock parameters have the same mean and variance (for the constant volatility (CV), GARCH(1,1) and Random Walk-GARCH (RW-GARCH) specifications).

2.3 Data

We use real-time data for the euro area for three variables, which we obtained from the real-time database of the ECB: the quarterly growth rate of GDP, inflation rate, nominal interest rates. We also use the five-year ahead inflation expectation as provided by the ECB survey of professional forecasters (SPF). The duration of the longest vintage is 1995Q2 – 2017Q2 while the duration of the shortest vintage is 1995Q2 – 2000Q4 (23 periods), with 67 vintages used overall. It should be noted that the time series for inflation expectations (see eq. 9) is shorter than for other variables and starts in 2000Q1 with two observations before that point.

3 Results

As referred to in the introduction section, several authors have used univariate or multivariate models or DSGE models to show that density forecasts can be improved upon by including time variation. Among these, Clarke (2007) demonstrates this point within a Bayesian VAR setup, using real time data for the US. The closest recent work to the present study is that of Diebold *et al.* (2017), where the authors show within a DSGE model that time variation in the form of SV improves density forecast performance, using US real time data. However, their approach is computationally expensive, making it difficult to extend their results to larger models (see e.g. Pitt *et al.* (2012) who point to increased computational costs for noisy posterior calculations). Here, we suggest an alternative form that is computationally cheaper, modeling time variation in the form of a GARCH process, and demonstrate its performance using EMU real time data.

Both Clark (2011) and Diebold *et al.* (2017) use variants of the Metropolis-within-Gibbs algorithm, which involves the use of the usual Metropolis-Hastings (MH) random walk algorithm for the parameters of the model, and Gibbs sampling for the stochastic volatility component (based on a Gaussian mixture approximation).

In contrast, our approach allows us to compute the "stochastic volatility component" of the law of motion by filter (without approximation) with increased computational speed. Moreover, we test our approach under less favorable conditions (in terms of estimation quality). We use posterior mode estimation instead of fully Bayesian estimation.

Additionally, the time period considered in Diebold *et al.* (2017), 1964Q2-2011Q2, encompasses the period before and after the great moderation - a period under which the stochastic volatility model faces more favorable conditions. They also use a structural break model, where a break happens during the great moderation, in order to analyze its performance against SV models. In contrast to their sample, our sample starts in 1995Q2.

In the following, we discuss and compare the point and density forecast performance of the three models that we use.

3.1 Point Forecasts

We use Dynare (Adjemian *et al.* (2011)) for the estimation of the model with real-time data. Our sample consists of 67 vintages of quarterly data for the euro area and spans the period 1995Q2 - 2017Q2 with the shortest vintage covering the period 1995Q2 – 2000Q4 (23 periods). We estimate the model on each data-file (including short vintages), using the four observed variables, as discussed in section 2.3. We estimate three versions of the model, i.e. CV, GARCH and RW-GARCH by maximum posterior and use the first four quarters for pre-sampling. Then forecasts are generated for the 1- to 8-step ahead horizons from each data-point.

Table 2 presents Root Mean Square Errors (RMSE) of point forecasts for the different models we estimate (we use the freshest data as the true one). Our results suggest that the GARCH specification outperforms all other specifications in forecasting inflation, output growth (except at the two quarters ahead horizon) and short-term forecasting of interest rates, while the CV model produces superior inflation expectations forecasts. The last result, that the CV model outperforms in the forecast of inflation expectations, is not surprising if one takes into account that inflation expectations data are less volatile in comparison to the other observed variables.

Table 2: RMSE for point forecasts

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV	0.56	0.81	0.91	0.96	0.99	1.01	1.03	1.04
	RW-GARCH	0.49	0.76	0.90	0.96	0.99	1.00	1.02	1.03
	GARCH	**0.48**	**0.74**	**0.89**	**0.96**	**0.98**	**0.99**	**1.00**	**1.01**
zzobs_dP40	CV	**0.06**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**	**0.11**	**0.11**
	RW-GARCH	0.06	0.07	0.08	0.09	0.11	0.12	0.13	0.14
	GARCH	0.06	0.07	0.09	0.11	0.13	0.14	0.16	0.17
zzobs_R	CV	0.45	0.79	1.07	1.29	1.47	**1.61**	**1.73**	**1.82**
	RW-GARCH	0.43	0.78	1.07	1.34	1.56	1.75	1.93	2.08
	GARCH	**0.41**	**0.73**	**1.01**	**1.26**	**1.47**	1.65	1.81	1.95
zzobs_dY	CV	0.81	**0.72**	0.77	0.80	0.79	0.83	0.84	0.85
	RW-GARCH	0.87	0.77	0.79	0.80	0.79	0.81	0.81	0.83
	GARCH	**0.78**	0.74	**0.75**	**0.75**	**0.75**	**0.76**	**0.76**	**0.77**

Because the errors of the shorter samples may affect the results, we use two measures for robustness of the quality of point forecasts. The first measure is a test which is based on Clarke (2007), where the H0 implies that two of the models that are compared have equal forecasting ability. The respective p-values for the test are presented in Table 3[3]. It is apparent from the results that there are significant advantages of using the GARCH specification for interest rate forecasts across all horizons and for output forecasts at longer horizons. In contrast, the CV model is superior in the forecast of inflation expectations.

[3] A p-value < 0.5 suggests an advantage for model 1 (with smaller number of parameters).

Table 3: P-Values for the test of equality of forecasting ability

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV vs. RW	**99.9%**	83.9%	64.6%	40.1%	**4.9%**	22.1%	18.3%	21.7%
	CV vs. GARCH	**99.9%**	69.0%	45.0%	15.7%	**1.5%**	10.0%	**2.6%**	5.9%
	RW vs. GARCH	59.8%	**95.9%**	55.0%	**3.8%**	**0.8%**	10.0%	12.3%	14.9%
zzobs_dP40	CV vs. RW	**0.1%**	31.0%	**1.6%**	**0.6%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	CV vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	RW vs. GARCH	6.8%	**0.1%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
zzobs_R	CV vs. RW	**100.0%**	**100.0%**	**99.6%**	89.6%	35.2%	15.3%	**4.6%**	**0.2%**
	CV vs. GARCH	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**99.8%**	90.0%	55.1%	14.9%
	RW vs. GARCH	**99.9%**	**99.9%**	**99.8%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**
zzobs_dY	CV vs. RW	93.2%	50.0%	73.4%	59.9%	18.7%	50.0%	55.1%	50.0%
	CV vs. GARCH	93.2%	83.9%	94.8%	77.5%	**95.1%**	**98.0%**	**99.7%**	**99.8%**
	RW vs. GARCH	59.8%	59.8%	**99.2%**	**97.9%**	**95.1%**	77.9%	74.1%	**98.2%**

To ensure that potentially large errors of the initial periods will not significantly affect our forecasting results, we also produce forecasts for the post-2009 period using the last 34 periods of our sample (i.e. 2009Q1 to 2017Q2), computing the RMSE and using the test of Clarke (2007) for model equality. As is apparent from Table 4, the RMSE results once again confirm that the GARCH model is superior across almost all horizons to other models for forecasts of the growth rate of GDP, inflation rate and nominal interest rates, while the CV model produces superior forecasts for long-term inflation expectations.

Table 4: RMSE (forecasts made using data range 2009Q1 – 2017Q2)

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV	0.53	0.75	0.88	0.97	1.04	1.11	1.16	1.21
	RW-GARCH	0.45	0.68	0.84	0.94	1.02	1.07	1.12	1.16
	GARCH	**0.42**	**0.62**	**0.77**	**0.87**	**0.95**	**1.01**	**1.07**	**1.10**
zzobs_dP40	CV	**0.06**	**0.07**	**0.09**	**0.10**	**0.11**	**0.13**	**0.14**	**0.14**
	RW-GARCH	0.06	0.08	0.10	0.12	0.14	0.15	0.17	0.18
	GARCH	0.07	0.09	0.12	0.14	0.17	0.19	0.21	0.22
zzobs_R	CV	0.43	0.77	1.03	1.25	1.43	1.58	1.70	1.81
	RW-GARCH	0.38	0.70	0.98	1.23	1.45	1.65	1.83	2.01
	GARCH	**0.33**	**0.60**	**0.84**	**1.05**	**1.24**	**1.41**	**1.58**	**1.73**
zzobs_dY	CV	0.58	**0.43**	0.47	0.51	0.54	0.56	0.53	0.57
	RW-GARCH	0.67	0.49	0.47	0.49	0.50	0.51	0.47	0.49
	GARCH	**0.52**	0.45	**0.43**	**0.45**	**0.47**	**0.48**	**0.46**	**0.47**

In Table 5, the p-values that correspond to Clarke (2007)'s test of equality of forecasts are presented when we use the period 2009Q1-2017Q2. The results suggest that the GARCH specification is significantly superior to other specifications across all horizons for interest rate forecasts and for the first four horizons of the inflation rate. Further, the results are mixed for output growth forecasts, i.e. the GARCH specification is superior to the RW specification but not to the CV specification. For forecasts of inflation expectations, the test once more suggests that the CV model is preferred over other models.

Table 5: P-values of RMSE-test (forecasts made using data range 2009Q1 – 2017Q2)

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV vs. RW	**99.9%**	**99.0%**	**98.5%**	57.2%	50.0%	71.4%	77.9%	91.6%
	CV vs. GARCH	**99.9%**	**99.0%**	**99.8%**	**95.1%**	64.4%	82.8%	77.9%	83.7%
	RW vs. GARCH	94.5%	**100.0%**	**100.0%**	**99.2%**	93.2%	90.8%	77.9%	83.7%
zzobs_dP40	CV vs. RW	5.5%	43.0%	23.7%	**4.9%**	**0.4%**	**0.0%**	**0.0%**	**0.0%**
	CV vs. GARCH	**0.4%**	**0.4%**	**0.2%**	**0.3%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	RW vs. GARCH	5.5%	**0.4%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
zzobs_R	CV vs. RW	**100.0%**	**100.0%**	92.5%	81.9%	35.6%	17.2%	**2.6%**	**0.0%**
	CV vs. GARCH	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**99.5%**
	RW vs. GARCH	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**
zzobs_dY	CV vs. RW	89.2%	5.5%	14.1%	10.0%	13.2%	42.5%	64.9%	83.7%
	CV vs. GARCH	81.1%	18.9%	36.0%	29.2%	35.6%	42.5%	77.9%	91.6%
	RW vs. GARCH	**97.5%**	89.2%	**100.0%**	**99.7%**	93.2%	57.5%	50.0%	**96.2%**

3.2 Density Forecasts

The point forecast exercise that was discussed in the previous subsection demonstrates that for inflation, interest and growth rates, time-varying models outperform CV models and further, that the GARCH specification is the superior time-varying model. In this subsection, we estimate our models to obtain density forecasts. As a measure of forecasting quality, we use the log predictive density score (LPS), which is a broad measure of calibration of density forecasts.

$$\begin{aligned} LPS(Y_{t+h}) = \log\, p(Y_{t+h}|I_t) &= -(n_Y/2)\log(2\pi) - \log\,(|V_t(Y_{t+h})|)/2 \\ &-0.5(Y_{t+h} - E_tY_{t+h})'(V_t(Y_{t+h}))^{-1}(Y_{t+h} - E_tY_{t+h}) \end{aligned} \tag{20}$$

Table 6 presents LPS values for out-of-sample forecasts. We note that for inflation expectations, interest rates (longer horizons) and output growth rate (shorter horizons), the CV specification is superior to other specifications. For forecasts of the inflation rate, the RW-GARCH specification is superior in the first six quarters whereas for the last two quarters the CV and GARCH specifications alternate in superiority. Also, the GARCH specification produces superior density forecasts in the first quarters of the interest rate and the last two quarters of the output growth rate.

In addition to univariate density forecasts, we examine the extent to which the results are affected by multivariate forecasting. In the two cases we analyze – when all four variables are included, and when inflation expectations are excluded - the CV specification dominates other specifications. This result is similar to Diebold *et al.* (2017)'s finding, that CV produces better forecasts in multivariate settings.

However, their ranking according to RMSE and LPS results are similar for each variable while our results for each variable are sensitive to the measure used, a result that is related to the higher influence of short sample.

Table 6: Log Predictive Scores (LPS)

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV	-0.85	-1.23	-1.39	-1.46	-1.48	-1.50	-1.52	**-1.54**
	RW-GARCH	**-0.68**	**-1.07**	**-1.23**	**-1.30**	**-1.34**	**-1.41**	-1.53	-1.69
	GARCH	-0.69	-1.08	-1.26	-1.36	-1.41	-1.45	**-1.52**	-1.58
zzobs_dP40	CV	**1.46**	**1.43**	**1.22**	**1.16**	**1.02**	**0.94**	**0.87**	**0.86**
	RW-GARCH	0.68	0.32	0.69	0.59	0.71	0.54	0.50	0.62
	GARCH	1.33	1.11	0.96	0.84	0.71	0.61	0.53	0.49
zzobs_R	CV	-0.68	-1.43	**-1.97**	**-2.37**	**-2.64**	**-2.84**	**-3.01**	**-3.13**
	RW-GARCH	-0.45	-1.37	-2.11	-2.68	-3.11	-3.41	-3.62	-3.76
	GARCH	**-0.43**	**-1.36**	-2.10	-2.84	-3.42	-3.80	-4.06	-4.30
zzobs_dY	CV	**-1.34**	**-1.32**	**-1.42**	**-1.46**	**-1.40**	**-1.41**	-1.41	-1.40
	RW-GARCH	-1.36	-1.55	-1.67	-1.74	-1.66	-1.58	-1.53	-1.51
	GARCH	-1.51	-1.70	-1.85	-1.84	-1.71	-1.53	**-1.39**	**-1.40**
all but zzobs_dP40	CV	-3.10	-3.98	**-4.90**	**-5.53**	**-5.87**	**-6.29**	**-6.71**	**-7.05**
	RW-GARCH	-2.72	**-3.90**	-5.04	-6.02	-6.51	-7.10	-7.79	-8.50
	GARCH	**-2.72**	-4.03	-5.36	-6.50	-7.13	-7.50	-7.92	-8.37
all (multivariate)	CV	**-1.71**	**-2.60**	**-3.79**	**-4.53**	**-5.08**	**-5.62**	**-6.17**	**-6.59**
	RW-GARCH	-4.46	-6.75	-6.38	-8.90	-8.37	-10.25	-11.46	-10.85
	GARCH	-2.23	-3.54	-4.87	-6.18	-6.99	-7.68	-8.39	-8.89

The LPS results in Table 6 suggest that the CV specification is superior in forecasting several variables in univariate and multivariate settings. Table 7 presents p-values associated with Clarke (2007)'s test of equal forecasting ability between the models we consider. In contrast to the LPS values presented in Table 6, the p-values indicate that the GARCH volatility model is superior across all horizons for forecasts of the interest rate and growth rate of GDP, and the first quarter of the inflation rate, though significance in the case of interest rates is given for the first four quarters (similar to the p-values of point forecasts that are presented in Table 3. For forecasts of the inflation rate, the GARCH specification produces superior forecasts only in the very short term while in the longer term the CV specification dominates. It is worth pointing out that, while RMSE and LPS values for point and density forecasts (Tables 2 and 6) deliver very dissimilar results, test for equality of forecasts (Tables 3 and 7) deliver very similar results.

Overall, we can see that the GARCH specification performs better than the CV specification, especially in short-term forecasting for most periods. Also, the GARCH specification performs better than the RW specification, while CV is mostly superior to the RW specification for longer horizons. In a multivariate setting, the RW specification is superior when inflation expectations are excluded, but the results are mixed when all variables are included (the GARCH specification is superior in the shorter horizons). In Table 7 we present p-values of forecast equality[4].

[4] Because a higher log predictive score indicates better forecasting performance for density forecasts while a lower RMSE indicates better forecasting quality for point forecasts, a value of p>0.5 indicates that model 1 is preferred (in contrast to p-values for the RMSE results in Table 6).

Table 7: P-values of forecast equality (for LPS)

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV vs. RW	**0.3%**	31.0%	55.0%	59.9%	45.0%	77.9%	81.7%	94.1%
	CV vs. GARCH	**0.3%**	31.0%	64.6%	40.1%	91.9%	90.0%	92.2%	**98.2%**
	RW vs. GARCH	6.8%	69.0%	87.0%	**98.9%**	**99.2%**	**99.0%**	**99.9%**	**96.6%**
zzobs_dP40	CV vs. RW	77.1%	**99.9%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
	CV vs. GARCH	93.2%	**99.9%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
	RW vs. GARCH	**99.9%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
zzobs_R	CV vs. RW	**0.0%**	**0.1%**	5.2%	30.7%	73.7%	84.7%	87.7%	94.1%
	CV vs. GARCH	**0.0%**	**0.1%**	**3.0%**	15.7%	26.3%	30.4%	74.1%	85.1%
	RW vs. GARCH	**1.2%**	**1.2%**	**0.8%**	**2.1%**	12.6%	30.4%	34.9%	39.7%
zzobs_dY	CV vs. RW	**1.2%**	**1.2%**	**0.8%**	**2.1%**	12.6%	22.1%	25.9%	21.7%
	CV vs. GARCH	**0.3%**	**0.6%**	**0.0%**	**0.0%**	**0.8%**	**0.2%**	**0.0%**	**0.1%**
	RW vs. GARCH	22.9%	**0.6%**	**0.8%**	**0.3%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
all but zzobs_dP40	CV vs. RW	**0.6%**	**0.1%**	**0.0%**	**0.1%**	8.1%	30.4%	25.9%	50.0%
	CV vs. GARCH	**0.0%**	**0.0%**	**0.1%**	**0.6%**	**1.5%**	10.0%	7.8%	30.1%
	RW vs. GARCH	**0.6%**	**4.1%**	5.2%	10.4%	**4.9%**	**3.6%**	**0.0%**	**0.0%**
all (multivariate)	CV vs. RW	**4.1%**	**2.3%**	26.6%	22.5%	73.7%	90.0%	87.7%	**98.2%**
	CV vs. GARCH	**0.6%**	10.7%	35.4%	40.1%	45.0%	69.6%	**95.4%**	**96.6%**
	RW vs. GARCH	10.7%	31.0%	45.0%	77.5%	64.8%	50.0%	65.1%	30.1%

Similar to the point forecast exercise in the previous subsection, we provide measures of density forecast quality (Table 8) and test for equality of the density forecast (Table 9) using post-2009 vintages. The results in Table 8 imply that the GARCH specification is significantly superior across all horizons for forecasts of the inflation rate, output growth rate, in the multivariate setting excluding inflation expectations and for the first quarters of the interest rate and the very short term of the multivariate forecast including all variables. In contrast, for quarters 5-8 of the interest rate, all horizons of inflation expectations and the longer term forecasts of the multivariate setting, the CV specification is superior. The RW-GARCH setting is inferior for all variables and horizons.

Table 8: LPS (forecasts made using data range 2009Q1 - 2017Q2)

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV	-0.83	-1.14	-1.29	-1.40	-1.50	-1.58	-1.66	-1.72
	RW-GARCH	-0.70	-1.10	-1.29	-1.40	-1.49	-1.55	-1.62	-1.66
	GARCH	**-0.68**	**-1.05**	**-1.23**	**-1.33**	**-1.41**	**-1.47**	**-1.53**	**-1.59**
zzobs_dP40	CV	**1.38**	**1.20**	**0.98**	**0.88**	**0.74**	**0.64**	**0.56**	**0.52**
	RW-GARCH	1.30	1.03	0.78	0.66	0.51	0.40	0.32	0.26
	GARCH	1.24	0.97	0.65	0.49	0.31	0.18	0.08	0.00
zzobs_R	CV	-0.57	-1.21	-1.62	-1.93	**-2.17**	**-2.33**	**-2.47**	**-2.57**
	RW-GARCH	-0.53	-1.27	-1.85	-2.30	-2.65	-2.91	-3.09	-3.24
	GARCH	**-0.37**	**-1.04**	**-1.52**	**-1.90**	-2.18	-2.40	-2.58	-2.72
zzobs_dY	CV	-1.09	-1.04	-1.07	-1.10	-1.12	-1.14	-1.14	-1.17
	RW-GARCH	-1.05	-1.05	-1.05	-1.09	-1.12	-1.14	-1.15	-1.19
	GARCH	**-0.93**	**-0.95**	**-0.93**	**-0.96**	**-0.98**	**-1.00**	**-1.01**	**-1.04**
all but zzobs_dP40	CV	-2.54	-3.72	-4.49	-4.91	-5.18	-5.34	-5.51	-5.63
	RW-GARCH	-2.41	-3.63	-4.60	-5.22	-5.68	-5.99	-6.26	-6.48
	GARCH	**-1.98**	**-3.28**	**-4.14**	**-4.70**	**-5.05**	**-5.28**	**-5.43**	**-5.57**
all (multivariate)	CV	-1.17	-2.52	**-3.57**	**-4.21**	**-4.77**	**-5.19**	**-5.57**	**-5.85**
	RW-GARCH	-1.15	-2.66	-4.04	-5.06	-5.91	-6.57	-7.03	-7.49
	GARCH	**-0.96**	**-2.40**	-3.70	-4.56	-5.30	-5.79	-6.16	-6.54

P-values for the test of equal forecasting when post-2009 data are used, which are presented in Table 9, corroborate the findings in Table 8. Specifically, the values imply that in both the univariate as well as multivariate settings, the GARCH specification is superior to other specifications for all variables and across all periods. While the results are significant for all horizons for forecasts of the interest rate, the growth rate of GDP and in the multivariate setting excluding inflation expectations, they are significant for around half of the results for other variables. As discussed previously, the CV specification significantly outperforms other specifications for forecasts of inflation expectations.

Table 9: P-values of LPS-test (forecasts made using data range 2009Q1 – 2017Q2)

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	CV vs. RW	**0.0%**	18.9%	50.0%	57.2%	13.2%	28.6%	22.1%	42.3%
	CV vs. GARCH	**0.4%**	**1.0%**	7.5%	**0.8%**	35.6%	28.6%	22.1%	42.3%
	RW vs. GARCH	18.9%	10.8%	**1.5%**	10.0%	22.9%	42.5%	35.1%	42.3%
zzobs_dP40	CV vs. RW	**97.5%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
	CV vs. GARCH	**97.5%**	**99.9%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
	RW vs. GARCH	81.1%	**97.5%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
zzobs_R	CV vs. RW	**1.0%**	18.9%	50.0%	90.0%	**98.8%**	**99.4%**	**99.9%**	**100%**
	CV vs. GARCH	**0.0%**	**0.0%**	**3.5%**	10.0%	22.9%	17.2%	35.1%	42.3%
	RW vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.1%**
zzobs_dY	CV vs. RW	**2.5%**	18.9%	7.5%	**4.9%**	13.2%	17.2%	35.1%	27.9%
	CV vs. GARCH	**0.0%**	**0.1%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.1%**
	RW vs. GARCH	**0.1%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
all but zzobs_dP40	CV vs. RW	5.5%	**1.0%**	**0.5%**	10.0%	77.1%	82.8%	87.6%	91.6%
	CV vs. GARCH	**0.1%**	**0.4%**	**0.0%**	**0.3%**	**0.4%**	**4.4%**	6.1%	42.3%
	RW vs. GARCH	**0.0%**	**0.1%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
all (multivariate)	CV vs. RW	18.9%	18.9%	36.0%	29.2%	50.0%	82.8%	77.9%	91.6%
	CV vs. GARCH	10.8%	5.5%	23.7%	10.0%	35.6%	57.5%	64.9%	72.1%
	RW vs. GARCH	**0.4%**	**1.0%**	7.5%	**4.9%**	13.2%	9.2%	12.4%	8.4%

3.3 Estimated volatility path

Figures 1-4 present the plots of the time-varying standard deviations for three of the four shocks[5] included in the baseline model for the two time-varying specifications we utilize (RW-GARCH and GARCH(1,1)). We provide the minimum, maximum, median and mean of the estimated standard deviations that correspond to the 67 vintages we utilize in our estimation.

[5] As explained previously, the monetary policy shock does not exhibit GARCH characteristics. As a result, the GARCH and RW-GARCH specifications are modeled to have a constant variance for the monetary policy shock (and zero correlations).

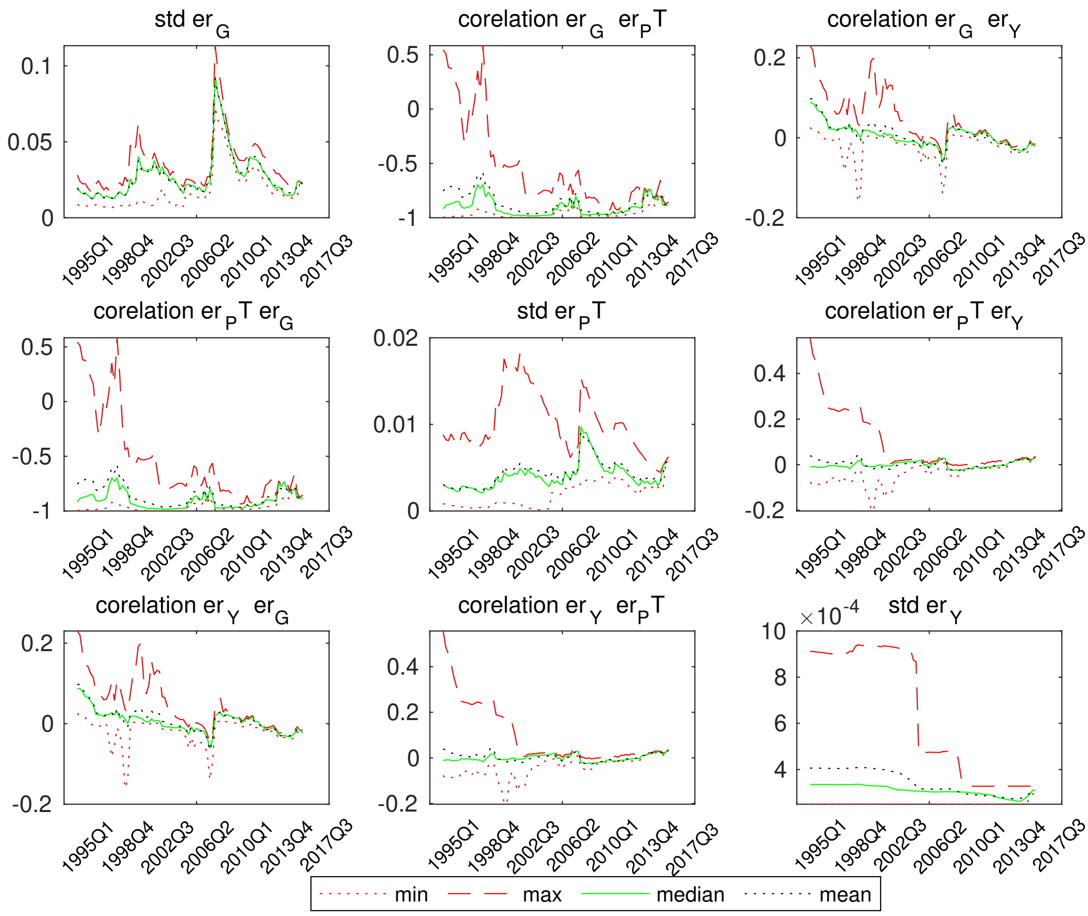

Figure 1. Volatility path generated by the RW-GARCH model.

Figure 1 displays the standard deviations and their correlations for the RW-GARCH model. It is apparent from the graphs that the shocks to government expenditure (ε_G) and the inflation target (ε_{PT}) exhibit substantial time-variation whereas variations in the shock to output growth are more muted. The graphs indicate that for the government expenditure and the inflation target shocks there were two peaks that correspond to the slowdown in 2001-2002 and the recession of 2008-2009 in the euro area. Correlations between government expenditure shocks on one side and output growth and the inflation target on the other side exhibit variation around the 2008 period but are mostly stable for the remaining periods. In contrast, the correlation between government expenditure and inflation target shocks exhibit substantial variation and are close to -1. Figure 2 shows the volatility paths that are based on the last 34 vintages.

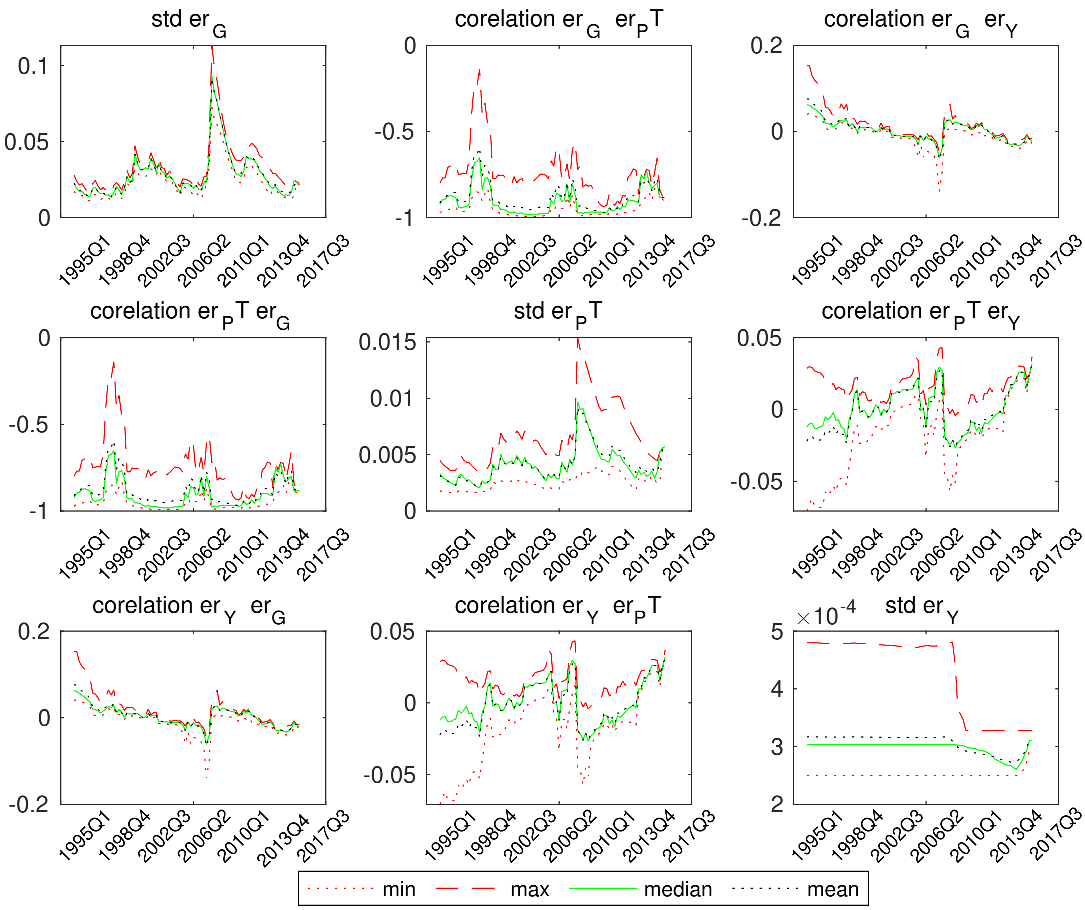

Figure 2. Volatility path generated by the RW-GARCH model (using data range 2009Q1 – 2017Q2).

Figure 2 displays the path of volatilities, as generated by the RW-GARCH model and using post-2009 vintages. It is apparent that there are large increases in the volatilities of shocks to the government expenditure and the inflation target that correspond to the 2008-2009 period. While there are mild fluctuations in the correlation of policy shocks with the TFP shock (ε_Y) during the 2008 crisis, the correlation between the two policy shocks (ε_G and ε_{PT}) fluctuates more significantly during this period.

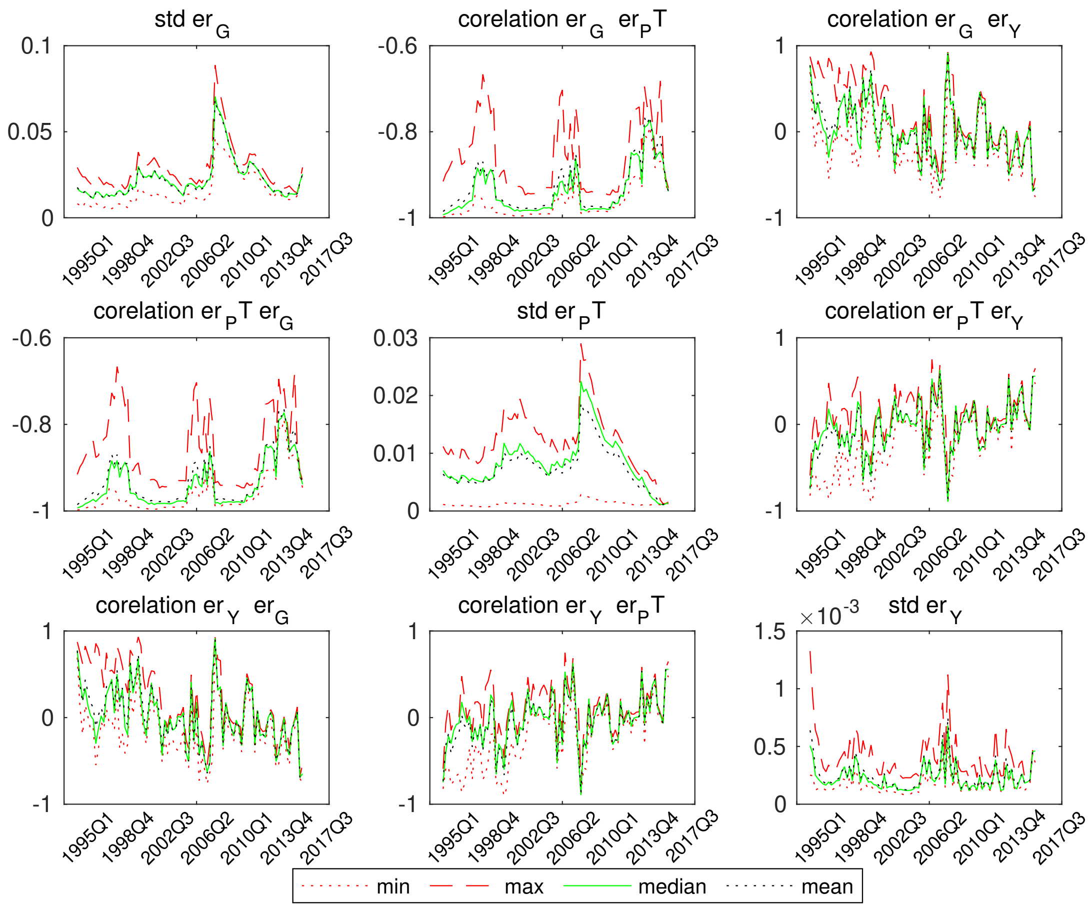

Figure 3. Volatility path generated by the GARCH(1,1) model.

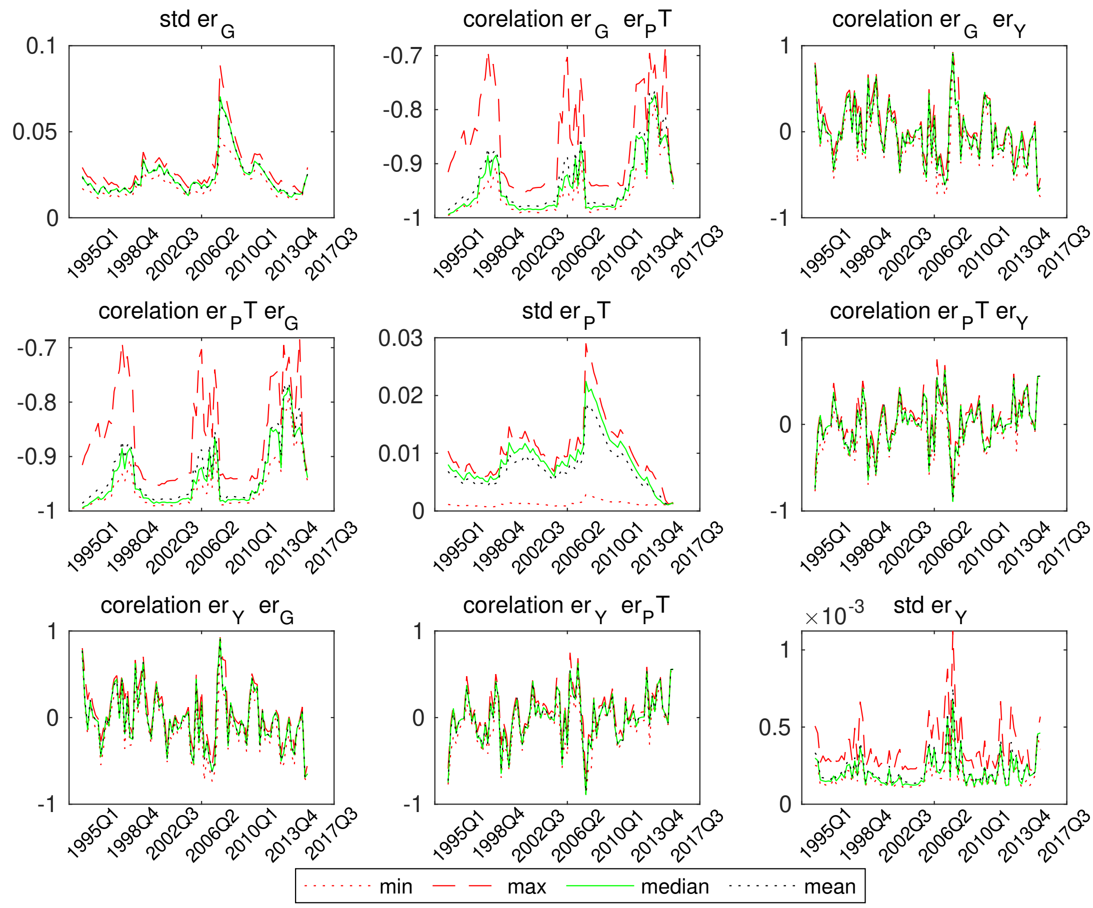

Figure 4. Volatility path generated by the GARCH(1,1) model (using data vintages 2009Q1 – 2017Q2).

Figures 3 and 4 display the paths of volatility as generated by the GARCH(1,1) model for the two data vintage periods considered, 1995-2017 and 2009-2017. The graphs indicate that the median results are mostly similar to those produced by the RW-GARCH model: there are two increases in the volatility that correspond to the period around 2001and the 2008 financial crisis. The increase during the financial crisis of 2008 is more substantial than the increase in the 2001 period.

3.4 Path of the Log Predictive Score

Figure 5 displays plots of 1-8 step ahead log predictive scores for all variables to account for out-of-sample forecasting performance of the various models we compare. The plots suggest that the GARCH model outperforms other models for most of the period and most of the steps and corroborates the results from the previous sections. While there are several periods when the CV specification is superior, the RW-GARCH specification is mostly inferior to the other specifications.

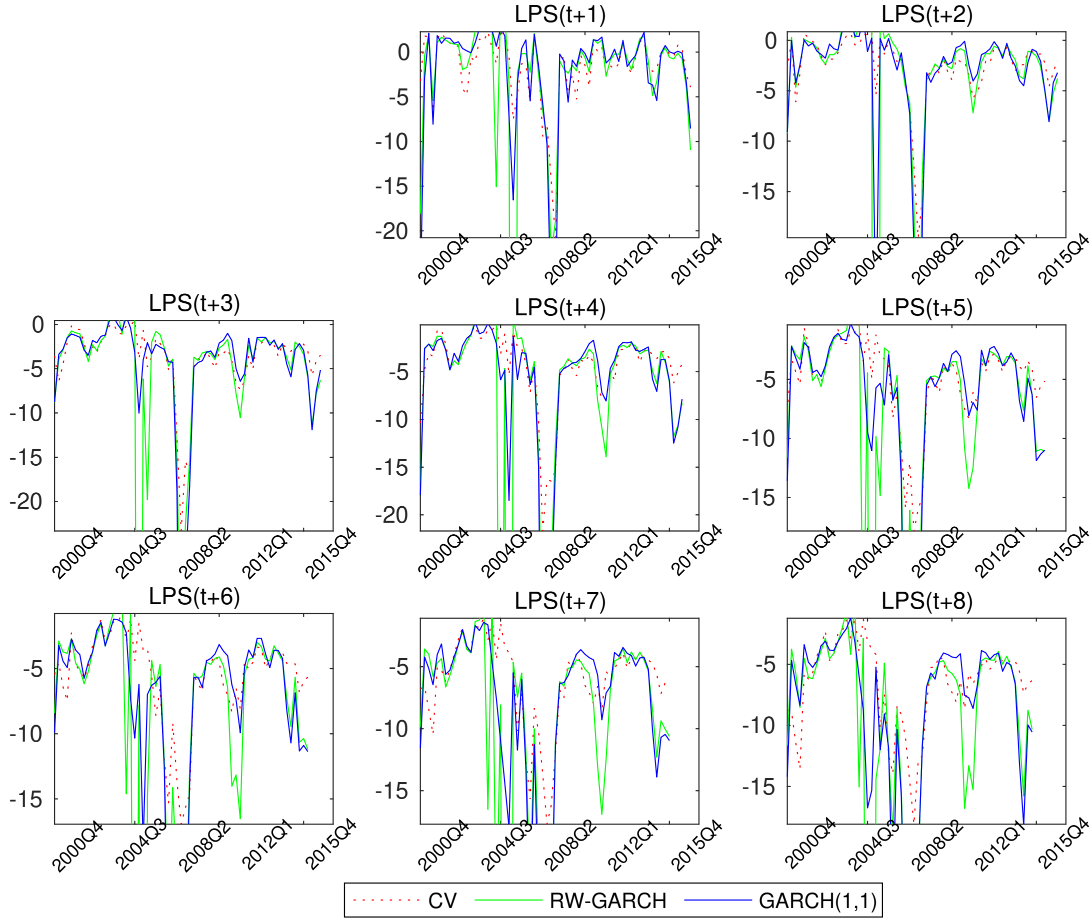

Figure 5. LPS for out-of-sample forecasts for all variables.

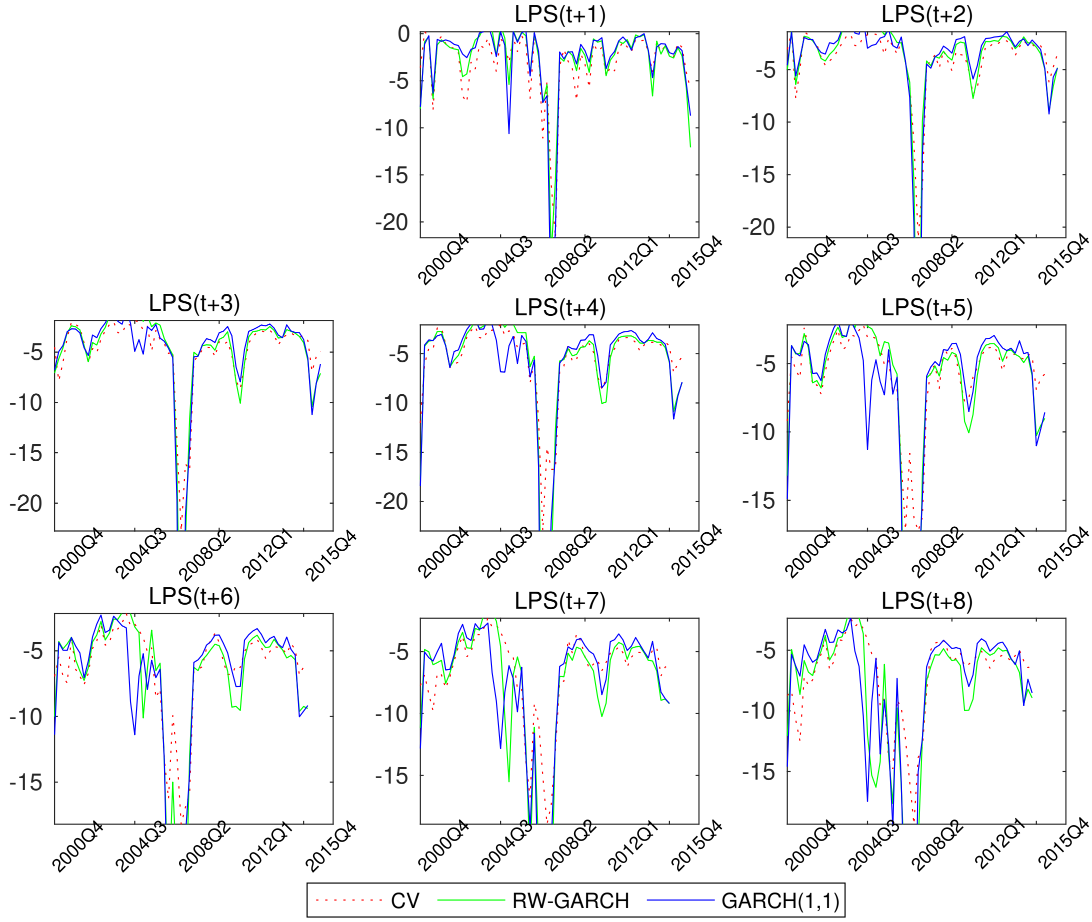

Figure 6. LPS for out-of sample forecasts for all variables except inflation expectations

Figure 6 displays the predictive scores when all variables are used for the out-of-sample forecasting performance test, with the exception of inflation expectations. The plots confirm once more the result that the GARCH specification outperforms other models. One notable difference is that, in contrast to the forecasting exercise when all variables are included (figure 5), the RW-GARCH specification produces forecasts that are superior to the CV specification for most periods.

3.5 In-sample forecast comparison with DSGE-SV

So far, we have compared the three specifications (GARCH, RW-GARCH and CV) and demonstrated that the GARCH specification produces mostly superior point and density forecasts within a DSGE model. In this subsection, it is our aim to compare the GARCH specification to the widely used stochastic volatility specification that was used in various influential studies such as Kim *et al.* (1998), Primiceri (2005), Justiniano & Primiceri (2008) or Diebold *et al.* (2017).

For this purpose, we produce in-sample point and density forecasts with both models and test for equality of the results, similar to the approach in the previous sections. In addition, we formulate two versions of the models, i.e. four versions in total: 1) GARCH with MCMC simulation, 2) GARCH with posterior mode estimation, 3) SV with an offset constant[6] of 10^{-4} and 4) SV with an offset constant of 10^{-8}.

Using the model introduced in section 2.2., the stochastic volatility process that governs

[6]This refers to the "offset constant" (see Kim *et al.* (1998)) that is added used in drawing stochastic volatilities and is explained more in detail in step 2 of the MCMC algorithm presented in the Appendix.

the volatilities of the DSGE-SV approach can be described as follows:

$$\log\left(E(\varepsilon_t\varepsilon_t'|V_t)_{i,i}\right) = \log(V_t)_{i,i} = h_{i,t} = h_i(1-\rho_i) + \rho_i h_{i,t-1} + \varepsilon_{v,i,t} \tag{21}$$

The algorithm uses an approximation of squared shocks by Gaussian mixture as follows

$$\log\left((\varepsilon_t\varepsilon_t')_{i,i}\right) \approx 2h_{i,t} + \varepsilon_{GM,t} \tag{22}$$

The steps of the MCMC algorithm used to characterize the posterior distribution of structural parameters are described in the Appendix[7].

Table 10 displays the results of the point forecast exercise. They suggest that in short horizons GARCH specifications are superior while in longer horizons forecasts generated by SV specifications are superior. Further, GARCH specifications dominate forecasts of inflation expectations and SV specifications dominate interest rate forecasts. The forecast errors generally don't deviate strongly from one model to another and there is also no pattern according to which one specification (GARCH with MCMC or GARCH with posterior maximization, and SV with offset coefficient 10^{-4} or 10^{-8}) is superior to the other.

Table 11 presents results of the density forecast exercise where we compare log predictive scores of the various models we utilize. These stay in stark contrast to the results in Table 10 and imply that GARCH specifications are superior to SV specifications for all variables that we consider, and in multivariate settings. While differences between the two GARCH specifications are mostly small, the GARCH specification with MCMC is superior to the GARCH specification with posterior maximization. The SV specifications exhibit significant sensitivity to the choice of the offset constant. Specifically, the specification with an offset constant of 10^{-4} is superior in the forecast of inflation expectations and interest rates while the specification with an offset constant of 10^{-8} is superior in the forecast of the inflation rate and output growth. In multivariate forecasting, the specification with an offset constant of 10^{-8} displays vastly inferior predictive quality. The latter result, that the SV model exhibits sensitivity to the choice of the offset parameter, is likely related to several factors such as the short sample and missing values of the inflation expectations data, and the choice of priors for the variance which implies that persistent volatility leads to permanently increasing expected variances due to exponential transformation.

Table 10: RMSE of in-sample forecasts

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	GARCH	0.46	0.70	**0.84**	**0.91**	0.95	0.97	0.98	0.99
	GARCH at mode	**0.45**	**0.70**	0.84	0.93	0.97	0.99	1.00	1.01
	SV (offset const. 10^{-4})	0.50	0.73	0.86	0.91	**0.93**	**0.95**	**0.97**	**0.97**
	SV (offset const. 10^{-8})	0.53	0.83	1.03	1.14	1.19	1.22	1.23	1.23
zzobs_dP40	GARCH	**0.06**	**0.06**	**0.08**	**0.09**	**0.10**	**0.11**	**0.12**	**0.12**
	GARCH at mode	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13
	SV (offset const. 10^{-4})	0.12	0.09	0.13	0.11	0.16	0.15	0.20	0.21
	SV (offset const. 10^{-8})	0.09	0.09	0.11	0.15	0.18	0.18	0.20	0.26
zzobs_R	GARCH	0.39	0.71	1.01	1.26	1.47	1.65	1.79	1.91
	GARCH at mode	0.39	0.70	0.99	1.23	1.43	1.60	1.73	1.84
	SV (offset const. 10^{-4})	0.43	0.75	1.00	1.20	**1.37**	**1.49**	**1.58**	**1.65**
	SV (offset const. 10^{-8})	**0.38**	**0.68**	**0.95**	**1.19**	1.40	1.58	1.71	1.80
zzobs_dY	GARCH	0.79	0.79	0.83	0.84	0.85	0.86	0.86	0.82
	GARCH at mode	**0.76**	**0.74**	0.76	0.77	0.78	0.79	0.78	0.74
	SV (offset const. 10^{-4})	0.83	0.76	**0.75**	**0.75**	**0.74**	**0.72**	**0.71**	**0.71**
	SV (offset const. 10^{-8})	0.96	0.89	0.86	0.84	0.85	0.84	0.85	0.79

[7]Our algorithm accounts for the correction in Del Negro & Primiceri (2015).

Table 11: LPS of in-sample forecasts

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	GARCH	-0.66	-1.05	-1.22	**-1.33**	**-1.38**	**-1.41**	**-1.44**	**-1.47**
	GARCH at mode	**-0.64**	**-1.03**	**-1.21**	-1.33	-1.40	-1.43	-1.46	-1.49
	SV (offset const. 10^{-4})	-4.12	-4.44	-4.64	-4.78	-4.91	-5.04	-5.17	-5.29
	SV (offset const. 10^{-8})	-1.15	-2.17	-2.86	-3.26	-3.45	-3.56	-3.56	-3.60
zzobs_dP40	GARCH	1.48	**1.26**	1.04	**0.93**	0.79	0.70	0.62	0.57
	GARCH at mode	**1.50**	1.26	**1.04**	0.92	**0.79**	**0.70**	**0.63**	**0.57**
	SV (offset const. 10^{-4})	-3.39	-3.85	-4.16	-4.41	-4.64	-4.84	-5.01	-5.18
	SV (offset const. 10^{-8})	-4.47	-4.64	-7.62	-9.84	-13.27	-15.58	-17.88	-19.01
zzobs_R	GARCH	**-0.35**	**-1.04**	**-1.49**	**-1.84**	**-2.07**	**-2.18**	**-2.25**	**-2.31**
	GARCH at mode	-0.38	-1.07	-1.51	-1.90	-2.16	-2.27	-2.34	-2.39
	SV (offset const. 10^{-4})	-4.57	-4.94	-5.17	-5.33	-5.47	-5.58	-5.67	-5.76
	SV (offset const. 10^{-8})	-8.33	-17.79	-25.85	-32.61	-38.13	-42.14	-44.80	-47.31
zzobs_dY	GARCH	**-1.07**	-1.19	-1.27	-1.37	-1.39	-1.34	-1.35	-1.36
	GARCH at mode	-1.07	**-1.12**	**-1.18**	**-1.26**	**-1.28**	**-1.22**	**-1.21**	**-1.24**
	SV (offset const. 10^{-4})	-4.72	-4.78	-4.85	-4.94	-5.03	-5.12	-5.22	-5.31
	SV (offset const. 10^{-8})	-3.05	-2.60	-2.49	-2.40	-2.42	-2.37	-2.39	-2.10
all but zzobs_dP40	GARCH	**-1.73**	-2.97	-3.73	**-4.30**	**-4.64**	**-4.81**	**-4.98**	**-5.12**
	GARCH at mode	-2.84	**-2.95**	**-3.69**	-4.30	-4.70	-4.87	-5.03	-5.19
	SV (offset const. 10^{-4})	-10.58	-13.02	-13.70	-14.19	-14.59	-14.92	-15.21	-15.45
	SV (offset const. 10^{-8})	-155.73	-189.46	-168.27	-151.57	-142.12	-131.83	-127.78	-125.45
all (multivariate)	GARCH	**-0.33**	**-1.84**	**-2.84**	**-3.51**	**-4.04**	**-4.25**	**-4.47**	**-4.64**
	GARCH at mode	-0.34	-1.84	-2.86	-3.60	-4.22	-4.48	-4.70	-4.89
	SV (offset const. 10^{-4})	-11.03	-13.88	-14.94	-15.76	-16.49	-17.11	-17.66	-18.17
	SV (offset const. 10^{-8})	-883.02	-598.15	-529.54	-476.80	-389.20	-331.43	-296.60	-262.71

Once more, we test for equality of forecasting as a robustness check using the test in Clarke (2007) [8]. Table 12 presents p-values[9] of the test associated with the RMSE results in Table 10 and imply that the values are mostly insignificant, that in the forecast of the inflation rate and inflation expectations and interest rates in the short term, the GARCH model produces superior forecasts, and that the SV model is dominant in the forecast of output growth and interest rates in the long term.

In contrast to this, the p-values associated with the LPS (Table 13) results show that the GARCH specification is superior for the forecast of all variables and in the multivariate environment. Finally, comparing the two GARCH specifications, the results reveal that both specifications alternate in superiority.

[8] We only report values for the SV model with an offset constant 10-4.

[9] A p-value < 0.5 suggests a smaller value of the statistic for model 1. GARCH and SV have the same number of parameters, so that the difference in parameters number does not affect the test statistic.

Table 12: P-values of in-sample RMSE-test

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	SV vs. GARCH	**98.8%**	66.6%	22.5%	58.6%	54.3%	50.0%	70.9%	50.0%
	SV vs. GARCHm	**99.6%**	74.0%	70.5%	66.8%	45.7%	41.3%	54.4%	25.3%
	GARCH vs. GARCHm	77.2%	85.8%	83.4%	66.8%	83.7%	81.0%	45.6%	41.2%
zzobs_dP40	SV vs. GARCH	**99.7%**	**99.9%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**
	SV vs. GARCHm	**99.7%**	**99.9%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**	**100.0%**
	GARCH vs. GARCHm	**0.0%**	**0.6%**	9.7%	**2.2%**	9.7%	**3.8%**	6.2%	14.4%
zzobs_R	SV vs. GARCH	94.5%	80.4%	70.5%	74.2%	45.7%	19.0%	11.2%	32.8%
	SV vs. GARCHm	88.0%	74.0%	70.5%	74.2%	54.3%	33.0%	22.0%	41.2%
	GARCH vs. GARCHm	22.8%	6.6%	5.3%	13.9%	22.3%	74.5%	63.0%	86.7%
zzobs_dY	SV vs. GARCH	29.7%	**2.7%**	16.6%	**2.5%**	**0.3%**	**1.4%**	**3.0%**	25.3%
	SV vs. GARCHm	37.5%	26.0%	29.5%	9.6%	**3.1%**	9.4%	**4.9%**	32.8%
	GARCH vs. GARCHm	70.3%	80.4%	**96.7%**	**95.9%**	94.9%	**99.9%**	**99.0%**	**97.8%**

Table 13: P-values of in-sample LPS-test

		$t+1$	$t+2$	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$	$t+8$
zzobs_dP	SV vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	SV vs. GARCHm	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	GARCH vs. GARCHm	**2.1%**	**2.7%**	8.0%	25.8%	37.2%	74.5%	63.0%	32.8%
zzobs_dP40	SV vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	SV vs. GARCHm	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	GARCH vs. GARCHm	63.8%	45.3%	54.7%	27.8%	36.2%	36.2%	36.2%	36.2%
zzobs_R	SV vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	SV vs. GARCHm	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	GARCH vs. GARCHm	45.8%	74.0%	92.0%	86.1%	**96.9%**	**97.6%**	**98.2%**	**99.3%**
zzobs_dY	SV vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	SV vs. GARCHm	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	GARCH vs. GARCHm	**2.1%**	**0.1%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
all but zzobs_dP40	SV vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	SV vs. GARCHm	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	GARCH vs. GARCHm	**0.2%**	**0.3%**	**3.3%**	25.8%	54.3%	25.5%	54.4%	67.2%
all (multivariate)	SV vs. GARCH	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	SV vs. GARCHm	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**	**0.0%**
	GARCH vs. GARCHm	9.7%	6.2%	**3.8%**	9.7%	79.5%	85.6%	**98.8%**	93.8%

While forecasting quality is the primary focus of the present study, we also compare the two models with regard to their computational speed. For this end we use the inefficiency factor (also called autocorrelation time), which is a common metric of computational efficiency, and measures the frequency of the number of draws that are larger than the number of independent draws[10]. Table 14 (in the Appendix) presents the results of the inefficiency factor calculations for the two models we compare. As is apparent, the performance of the DSGE-GARCH model is superior to that of the DSGE-SV model across all chains.

[10]The Appendix contains details of the computation of the inefficiency factor.

4 Conclusion

A number of authors suggested efficient procedures for incorporating time-varying volatilities into structural models. In this work, we have suggested a GARCH based approach to the modeling of time-varying volatility for DSGE models. Using a real-time dataset for the euro area, we show that this form of time-variation allows for a simple but powerful estimation procedure. To demonstrate this point we examine the forecasting performance of the DSGE-GARCH model against other benchmarks and show that our suggested model is mostly superior in terms of point and density forecasts.

An additional result relates to conditional correlations that are produced by the DSGE-GARCH approach. We can observe that changes of volatility near crisis periods are similar to the usual stochastic volatility approach. However, we observe changes of conditional correlations of shocks at such periods. We also observe a large correlation between policy shocks.

References

Adjemian, S., Bastani, H., Juillard, M., Mihoubi, F., Perendia, G., Ratto, M. & Villemot, S. 2011. Dynare: Reference manual, version 4.

Adolfson, M., Laséen, S., Lindé, J. & Svensson, L.E. 2011. Optimal monetary policy in an operational medium-sized dsge model. *Journal of Money, credit and Banking*, 43(7):1287–1331.

Andreasen, M.M. 2012. On the effects of rare disasters and uncertainty shocks for risk premia in non-linear dsge models. *Review of Economic Dynamics*, 15(3):295–316.

Blanchard, O.J. & Kahn, C.M. 1980. The solution of linear difference models under rational expectations. *Econometrica: Journal of the Econometric Society*, pages 1305–1311.

Bloom, N. 2009. The impact of uncertainty shocks. *econometrica*, 77(3):623–685.

Clark, T.E. 2011. Real-time density forecasts from bayesian vector autoregressions with stochastic volatility. *Journal of Business & Economic Statistics*, 29(3):327–341.

Clarke, K.A. 2007. A simple distribution-free test for nonnested model selection. *Political Analysis*, 15(03):347–363.

Collard, F. & Juillard, M. 2001. Accuracy of stochastic perturbation methods: The case of asset pricing models. *Journal of Economic Dynamics and Control*, 25(6-7):979–999.

Cúrdia, V. & Reis, R. 2010. Correlated disturbances and us business cycles. Technical Report, National Bureau of Economic Research.

Del Negro, M. & Primiceri, G.E. 2015. Time varying structural vector autoregressions and monetary policy: a corrigendum. *The review of economic studies*, 82(4):1342–1345.

Del Negro, M. & Schorfheide, F. 2013. Dsge model-based forecasting. In *Handbook of economic forecasting*, volume 2, pages 57–140. Elsevier.

Diebold, F.X., Schorfheide, F. & Shin, M. 2017. Real-time forecast evaluation of dsge models with stochastic volatility. *Journal of Econometrics*, 201(2):322–332.

Falter, A., Wesselbaum, D. *et al.*. 2018. Correlated shocks in estimated dsge models. *Economics Bulletin*, 38(4):2026–2036.

Fernández-Villaverde, J. & Rubio-Ramírez, J. 2010. Macroeconomics and volatility: Data, models, and estimation. Technical Report, National Bureau of Economic Research.

Georgiadis, G. & Jančoková, M. 2020. Financial globalisation, monetary policy spillovers and macro-modelling: Tales from 1001 shocks. *Journal of Economic Dynamics and Control*, 121:104025.

Ivashchenko, S., Çekin, S.E., Kotzé, K. & Gupta, R. 2020. Forecasting with second-order approximations and markov-switching dsge models. *Computational Economics*, 56(4):747–771.

Justiniano, A. & Primiceri, G.E. 2008. The time-varying volatility of macroeconomic fluctuations. *American Economic Review*, 98(3):604–41.

Kim, S., Shephard, N. & Chib, S. 1998. Stochastic volatility: likelihood inference and comparison with arch models. *The review of economic studies*, 65(3):361–393.

Lucas, R. 1976. Econometric policy evaluation: a critique, w: The phillips curve and labor mar kets,(red.: K. brunner and ah meltzer).

Pichler, P. 2008. Forecasting with dsge models: The role of nonlinearities. *The BE Journal of Macroeconomics*, 8(1).

Pitt, M.K., dos Santos Silva, R., Giordani, P. & Kohn, R. 2012. On some properties of markov chain monte carlo simulation methods based on the particle filter. *Journal of Econometrics*, 171(2):134–151.

Primiceri, G.E. 2005. Time varying structural vector autoregressions and monetary policy. *The Review of Economic Studies*, 72(3):821–852.

Rubaszek, M. & Skrzypczyński, P. 2008. On the forecasting performance of a small-scale dsge model. *International Journal of Forecasting*, 24(3):498–512.

Schmitt-Grohé, S. & Uribe, M. 2004. Solving dynamic general equilibrium models using a second-order approximation to the policy function. *Journal of economic dynamics and control*, 28(4):755–775.

Sims, C.A. 2002. Solving linear rational expectations models. *Computational economics*, 20(1-2):1.

Sims, C.A. & Zha, T. 2006. Were there regime switches in us monetary policy? *American Economic Review*, 96(1):54–81.

Smets, F. & Wouters, R. 2007. Shocks and frictions in us business cycles: A bayesian dsge approach. *American economic review*, 97(3):586–606.

Appendix A

A.1 The Stochastic Volatility (SV) Model

In the previous sections, we demonstrated that specifying the volatility component of the model as a GARCH(1,1) process results in forecasting performance that is mostly superior to constant volatility or RW-GARCH specifications. In this section, we compare the performance of the GARCH(1,1) specification to the stochastic volatility specification (DSGE-SV), variants of which were introduced and used in several influential studies such as Kim *et al.* (1998), Justiniano & Primiceri (2008) and Diebold *et al.* (2017). In the following, we describe the algorithm that describes this approach and demonstrate that the DSGE-GARCH approach is more efficient than the DSGE-SV approach. The discussion of the algorithm closely follows Primiceri (2005) and Justiniano & Primiceri (2008)[11]. Since the priors we use are different, we make additional minor modifications to the algorithm.

A.1.1 The DSGE-SV Algorithm

The stochastic volatility process of the DSGE-SV approach can be described as follows:

$$\log\big(E(\varepsilon_t\varepsilon_t'|V_t)_{i,i}\big) = \log\big(V_t\big)_{i,i} = h_{i,t} = h_i(1-\rho_i) + \rho_i h_{i,t-1} + \varepsilon_{v,i,t} \tag{A1}$$

The algorithm uses approximation of squared shocks by Gaussian mixture (eq. A2).

$$\log\big((\varepsilon_t\varepsilon_t')_{i,i}\big) \approx 2h_{i,t} + \varepsilon_{GM,t} \tag{A2}$$

The MCMC algorithm used to characterize the posterior distribution of structural parameters is described in the following steps:

Step 0. Initialization. Define initial values of the main parameters (θ_0), variance related parameters ($\theta_{v,0}$), variance trajectory matrix ($\{V\}_0$), regime trajectory of the Gaussian mixture approximation ($\{R\}_0$) and shocks' trajectory ($\{\varepsilon\}_0$).

Step 1. Draw the variance trajectory ($\{V\}_j|\{\varepsilon\}_{j-1},\{R\}_{j-1},\theta_{v,j-1}$). Here the fact is utilized that conditional on the regime, the Gaussian mixture system in A1-A2 is linear Gaussian. Thus, the usual Kalman filter and draw approach can be used.

Step 2. Draw the structural shocks ($\{\varepsilon\}_j|\{V\}_j,\theta_{j-1},Y_t$). The solution approximation and equation for observed variables are linear, so that the usual Kalman filter and draw approach can be used. However, one difficulty with this step is that the squared shocks in eq. A2 can be too small so that their logarithm goes towards negative infinity. Also, in contrast to the conventional[12] use of an "offset constant" of the size 0.001, we use the maximum of squared shocks and 10^{-8} because an offset constant of 0.001 can be too large when the magnitude of the variances are in the region of 10^{-6} to 10^{-4}.

Step 3. Draw the indicators of the mixture approximations ($\{R\}_{j-1}|\{\varepsilon\}_j,\{V\}_j$).

Step 4. Draw the parameters of the model ($\theta_{try},\theta_{v,try}|\theta_{j-1},\theta_{v,j-1}$) [13].

Step 5. Accept or reject draws the DSGE parameters. $\theta_{try},\theta_{v,try}$ is accepted with probability α_θ (i.e., $\theta_j = \theta_{try}$, $\theta_{v,j} = \theta_{v,try}$), otherwise $\theta_j = \theta_{j-1}, \theta_{v,j} = \theta_{v,j-1}$. The acceptance probability a_θ depends on the prior distribution and the likelihood (computed with the Kalman filter). For volatility parameters the scheme is similar, but the computation of the likelihood is simpler for the model presented in eq. A1.

[11] Our algorithm accounts for the correction in Del Negro & Primiceri (2015).

[12] See e.g. Justiniano & Primiceri (2008). We demonstrate the sensitivity to this coefficient.

[13] Due to our use of different priors, this step deviates from the approach of Justiniano & Primiceri (2008), who draw the coefficients directly. Instead, we use a random walk to draw the coefficients of the stochastic volatilities.

$$a_\theta = \min\left\{1; \frac{p\big(\{Y\}|\theta_{try}, \theta_{v,try}, \{V\}_j\big)p\big(\theta_{try}, \theta_{v,try}\big)}{p\big(\{Y\}|\theta_{j-1}, \theta_{v,j-1}, \{V\}_j\big)p\big(\theta_{j-1}, \theta_{v,j-1}\big)}\right\} \tag{A3}$$

A.2 Comparison of Stochastic Volatility and GARCH

In order to compare the DSGE-SV and DSGE-GARCH approaches, we need to define the relation between their parameters. These two time-varying volatility specifications are very different and the relation between the parameters can be defined, using the one-dimensional case with the following approach: we start by using the two coefficients that are equal to each other, η_s^* (from the GARCH specification) and ρ^* (from the SV specification), where one is the coefficient associated with the lag of log-volatility, while the other is the coefficient related to the lag of expected volatility.

The second group of coefficients is related to the variance of volatility shocks ($\mathrm{var}(\varepsilon_{v^*,t})$). Equation A4 describes how the conditional volatilities are related. It should be noted that the GARCH and SV specifications have different timings. We divide the different expected variance terms for the GARCH and SV specifications to relate them such that they are independent of the previous variance.

$$\frac{\mathrm{var}_{t-1}\big(V_{i,i,t}\big)}{\big(E_{t-1}(V_{i,i,t})\big)^2} \underset{\text{SV definition}}{=} \frac{\Big(e^{16\mathrm{var}(\varepsilon_{v,i,t}/2} - e^{2(4\mathrm{var}(\varepsilon_{v,i,t})/2)}\Big)e^{4(h_i(1-\rho_i)+\rho_i h_{i,t-1})}}{e^{4(h_i(1-\rho_1)+\rho_i h_{i,t-1})+2(4\mathrm{var}(\varepsilon_{v,i,t})/2)}} =$$

$$= \frac{2(V_{t-2})^2(\eta_{s,i}\eta_{r,i})^4}{(V_{t-2})^2} \underset{\text{GARCH definition}}{=} \frac{\mathrm{var}_{t-2}(V_{t-1})}{(V_{t-2})^2} \tag{A4}$$

The last relation is for the unconditional variance (eq. A5), which is defined by the following in the GARCH approach,

$$E(V_{i,i,t}) \underset{\text{SV definition}}{=} e^{2h_i+4\mathrm{var}(\varepsilon_{v,i,t})/(1-(\rho_i)^2)/2} \tag{A5}$$

An important detail related to the DSGE-SV approach is the initialization of the Kalman filter: conventionally, the unconditional variance of model variables is used to initialize the filter but we use the unconditional mean variance of shocks, and compute the variance of model variables in the usual way. It may differ from the accurate unconditional variance of the model's variables.

We first present the sensitivity of volatility draws on the hidden parameter in step 2 of the stochastic volatility algorithm. We compute the mean volatility trajectory and corresponding standard deviations with fixed parameters and three values for the maximum squared shocks, 10^{-2}, 10^{-4} (our default) and 10^{-8}. The corresponding results are presented in figure 7. We present the trajectories in logarithms for better visual representation.

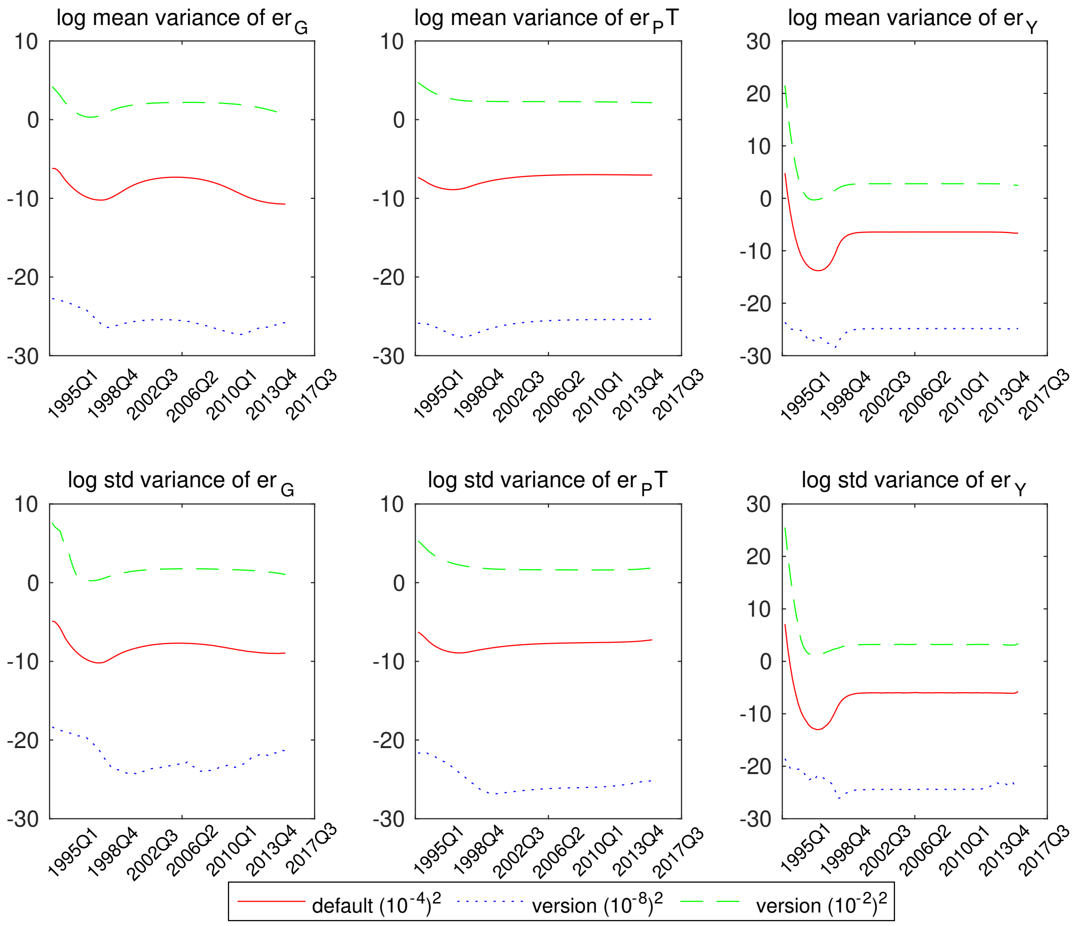

Figure 7. Volatility draws based on various hidden parameter choices.

We calculate two versions of variance trajectories in the DSGE-SV model: 1) conditional on the mode of the DSGE-GARCH model variables (i.e. with fixed parameters), and 2) joint draws of volatility and parameters (i.e. non-fixed parameters). To initialize the procedure for the fixed parameters specification, we use the posterior mode of DSGE-GARCH (using the longest vintage) and use the unconditional mean of the variance as the initial value. We then draw the stochastic volatilities, shocks and regime trajectories and compute 10 chains with 20 000 draws, drawn from the posterior distribution and compute the mean of the variance trajectory (without discarding draws). Following this we compute additional 10 chains with 20 000 draws using the mean of the variance trajectory to initialize the procedure. This results in the mean variance trajectory for the model with fixed parameters, as displayed in figure 8 (SV at GARCH-mode).

For the version with non-fixed parameters, we use the posterior mode of the DSGE-GARCH model as the initial value of parameters and use a Hessian-based approximation for the drawing procedure (step 4 of the SV algorithm), manually setting multipliers to have a conventional acceptance rate. To initialize the procedure we use the mean of the variance trajectory from the model with fixed parameters and compute 10 chains, with 20 000 draws for each chain. We compute the mean of the variance trajectory and variance of parameters, taking for each chain the last draw of parameters and dropping 25% of draws of each chain for the computation of the variance. We use the remaining draws as initial values for 10 additional chains with 20 000 draws for each chain. This results in the mean variance trajectory that is presented in figure 8 (SV-mean).

Figure 8 presents variance trajectories according to DSGE-GARCH (at the mode and mean) and DSGE-SV (SV at GARCH-mode and SV-mean) specifications for government spending (er_G), the inflation target (er_{PT}) and TFP shock (er_Y). The trajectories of variances calculated by the DSGE-SV models are close to each other and are smooth, with the exception of the TFP shock, where difference between the mode and the mean is substantial. The variances implied by the GARCH specifications show more substantial movements in comparison to the trajectories calculated by the DSGE-SV models and are lower for the inflation target (er_{PT}) and TFP shock (er_Y).

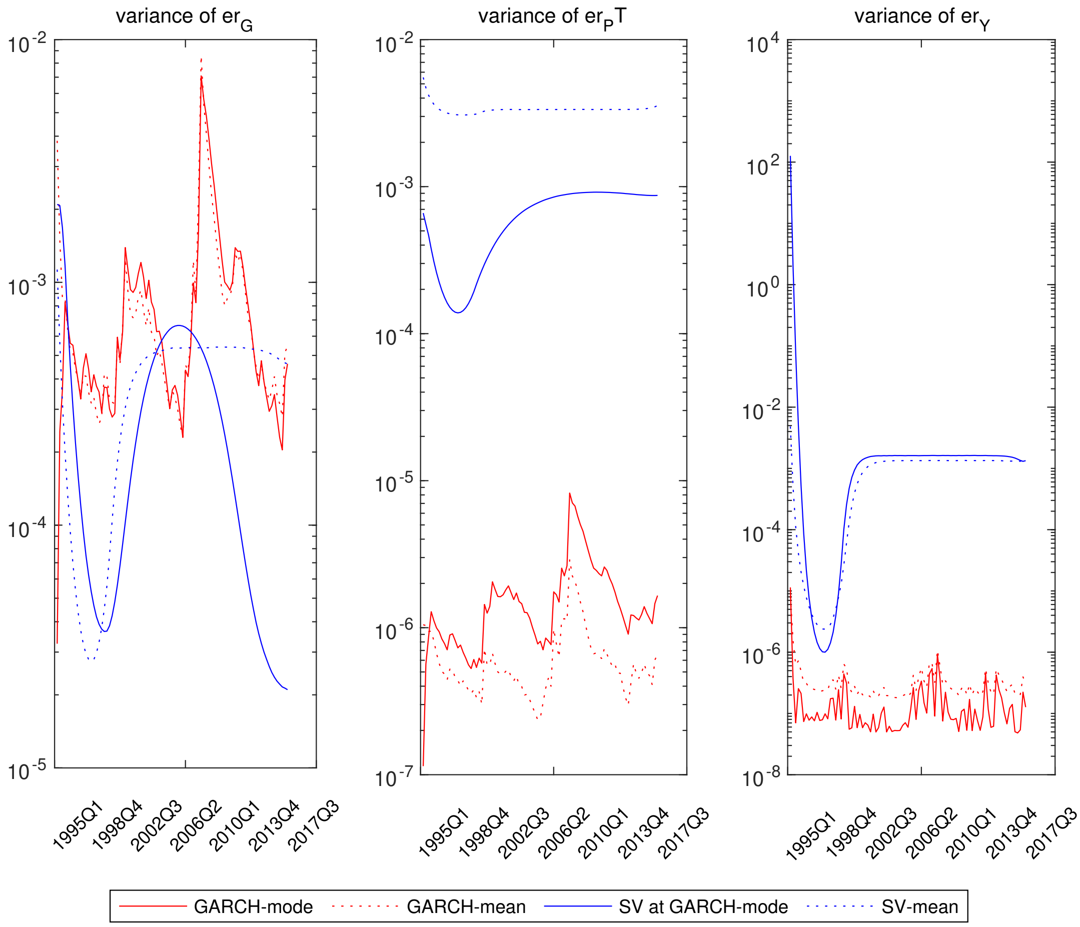

Figure 8. Variance according to DSGE-GARCH and DSGE-SV.

Figure 9 presents particular draws of the variances generated by the DSGE-SV specification and contrasts them to the averaged out variance trajectories that are generated by the DSGE-SV model. It's apparent that particular draws show more nuanced movements but their respective averages are more smooth.

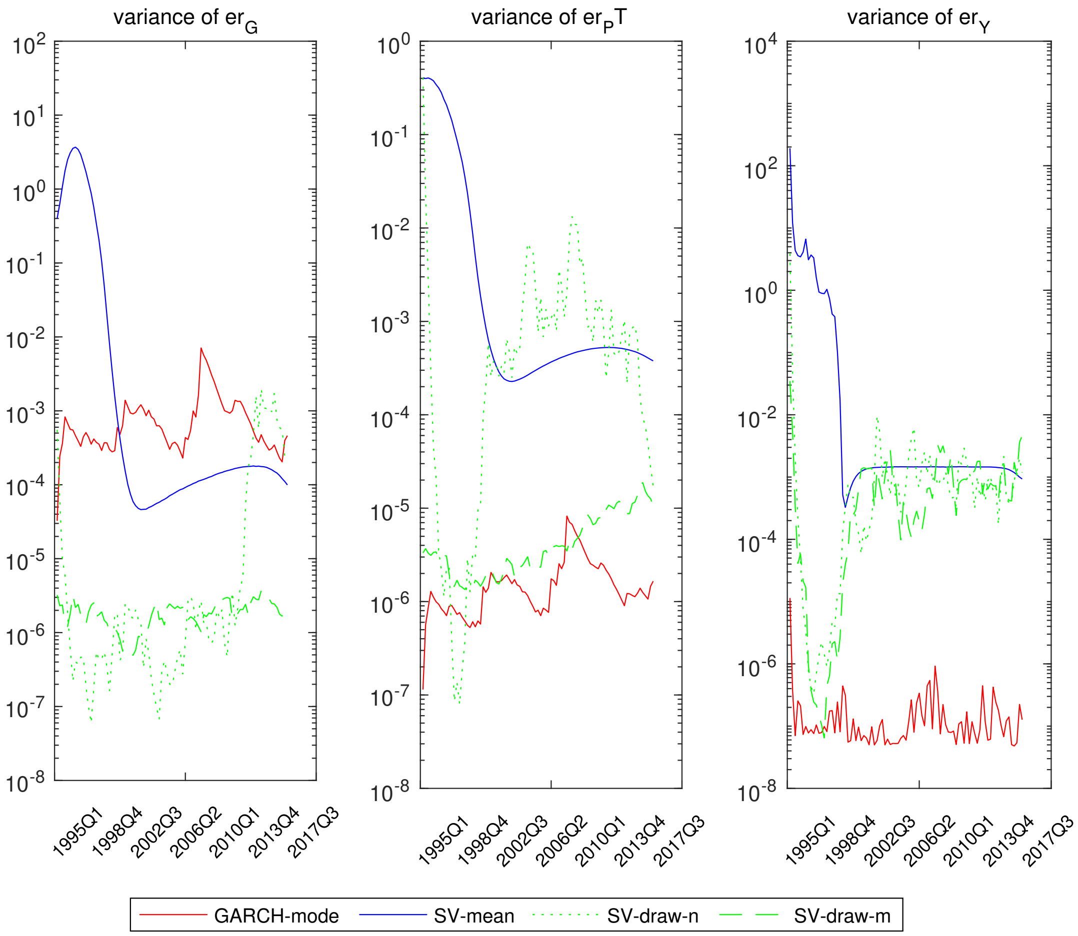

Figure 9. Variance according to DSGE-GARCH and particular draws of DSGE-SV.

A.2.1 Computational Efficiency

Finally, we compare the computational speed of the two approaches. Because the DSGE-SV approach uses an MCMC algorithm, the comparison of algorithms requires comparison of convergence speed in addition to direct comparison of computational costs. One such measure is the inefficiency factor which shows how many times the number of draws (correlated from MCMC) should be larger than the number of independent draws for the same variance of the estimated mean. The theoretical formula (A6) requires knowledge of all auto-correlations. Direct computation is inaccurate or impossible (when the sample is shorter than the number of lags) in case of highly persistence time series. However, the MH MCMC implies that density (of the full vector) conditional on first k-lags is equal to the density conditional on the first lag. It means that the infinite sum of the auto-covariance can be computed according to (A7) that is based on a VAR(1) presentation (with matrix A).

$$\mathrm{IF} = 1 + 2\sum_{i=1}^{\infty} \mathrm{correl}(x_{t+i}, x_t) \tag{A6}$$

$$\sum_{i=1}^{\infty} \mathrm{cov}(x_{k,t+i}, x_{k,t}) = \sum_{i=1}^{\infty} \mathrm{cov}(x_{t+i}, x_t) = \sum_{i=1}^{\infty} A^i \mathrm{var}(x_t) = A(I-A)^{-1}\mathrm{var}(x_t) \tag{A7}$$

The DSGE-SV is a Metropolis within Gibbs approach, which implies that model parameters and additional variables (volatility path, shocks, regimes) are described by Markov chains. However, a VAR(p) for model parameters can be used as a good approximation for all auto-covariances and we use a VAR(4) for the computation of the inefficiency factor[14]. Some statistics about the inefficiency factor are presented in Table 14. We have the following conventional acceptance rates: 20.97% (min across chains), 23.90% (mean across chains), 29.41% (max across chains) for DSGE-GARCH and 13.29% (min across chains), 29.57% (mean across chains), 37.45% (max across chains) for DSGE-SV. It should be noted that a negative inefficiency factor refers to the existence of an eigenvalue (of VAR representation) greater than one in modulus which implies an explosive process.

The table further displays the maximum, minimum, mean and median values of the inefficiency factor for the various chains for the DSGE-SV and DSGE-GARCH models and implies that the DSGE-GARCH model is superior across all chains with regard to computational speed and computational efficiency[15].

Table 14: Inefficiency factor

	DSGE-SV				DSGE-GARCH			
	min	median	mean	max	min	median	mean	max
chain 1	4.01E+3	7.93E+4	8.28E+4	1.66E+5	1.61E+2	1.96E+2	2.17E+2	3.56E+2
chain 2	-7.40E+5	-5.79E+4	-1.38E+5	4.86E+5	1.54E+2	2.12E+2	2.11E+2	3.58E+2
chain 3	-2.45E+4	2.43E+3	1.13E+3	1.76E+4	1.66E+2	2.42E+2	2.87E+2	8.30E+2
chain 4	8.67E+2	1.04E+4	1.11E+4	2.91E+4	1.64E+2	2.00E+2	2.07E+2	3.55E+2
chain 5	1.80E+2	6.40E+3	6.42E+3	1.35E+4	2.63E+2	2.90E+3	3.95E+3	1.46E+4
chain 6	4.99E+2	2.83E+3	3.46E+3	8.20E+3	1.46E+2	3.37E+2	4.03E+2	1.27E+3
chain 7	2.24E+3	5.38E+4	5.62E+4	1.33E+5	1.59E+2	2.53E+2	2.77E+2	5.26E+2
chain 8	3.75E+2	5.83E+3	7.45E+3	2.50E+4	1.52E+2	2.18E+2	2.73E+2	6.82E+2
chain 9	7.92E+2	6.25E+3	1.18E+4	3.24E+4	1.47E+2	1.94E+2	2.45E+2	8.61E+2
chain 10	6.34E+1	4.17E+3	5.50E+3	1.60E+4	1.77E+2	3.25E+2	4.39E+2	2.15E+3
min	-7.40E+5	-5.79E+4	-1.38E+5	8.20E+3	1.46E+2	1.94E+2	2.07E+2	3.55E+2
median	4.37E+2	6.04E+3	6.94E+3	2.70E+4	1.60E+2	2.30E+2	2.75E+2	7.56E+2
mean	-7.55E+4	1.13E+4	4.82E+3	9.27E+4	1.69E+2	5.07E+2	6.50E+2	2.20E+3
max	4.01E+3	7.93E+4	8.28E+4	4.86E+5	2.63E+2	2.90E+3	3.95E+3	1.46E+4

[14]It takes 45.27 seconds per 2 chains with 1000 draws each (plus default Dynare output) of MCMC DSGE-GARCH and 119.51 second per 2 chain 1000 draws each (without timing for output). Computations are made with CPU intel core i7-4720HQ 2.6GHz under Windows 8.

[15]The advantage of DSGE-GARCH is about 42.1 times according to mean (across chains) max (across parameters) of the inefficiency factor in addition to being faster by a measure of 2.6 in direct computational speed. For chains 2 and 3, the computational advantage becomes infinite, because two chains of DSGE-SV have explosive eigenvalues in their VAR(4) representation.